CS 516—Software Foundations via Formal Languages—Spring 2022

Problem Set 1

Model Answers

Problem 1

(a) To show that $\emptyset \to X = \{\emptyset\}$, we show that each side is a subset of the other.

Suppose $f \in \emptyset \to X$, so that f is a function, $\mathbf{domain}\, f = \emptyset$ and $\mathbf{range}\, f \subseteq X$. Because $\mathbf{domain}\, f = \emptyset$, we have that $f = \emptyset$. Thus $f \in \{\emptyset\}$.

Suppose $f \in \{\emptyset\}$, so that $f = \emptyset$. Then f is a function, $\mathbf{domain}\, f = \emptyset$ and $\mathbf{range}\, f = \emptyset \subseteq X$. Thus $f \in \emptyset \to X$.

(b) To show that $X \to \emptyset = \emptyset$, we show that each side is a subset of the other.

Suppose $f \in X \to \emptyset$, so that f is a function, $\mathbf{domain}\, f = X$ and $\mathbf{range}\, f \subseteq \emptyset$. Consequently $\mathbf{range}\, f = \emptyset$. Because $x \in X = \mathbf{domain}\, f$, there is a y such that $(x, y) \in f$. But then $y \in \mathbf{range}\, f = \emptyset$—contradiction. Thus we can conclude anything, including that $f \in \emptyset$.

And clearly $\emptyset \subseteq X \to \emptyset$.

(c) To show that $\{x\} \to X = \{\, \{(x, y)\} \mid y \in X \,\}$, we show that each side is a subset of the other.

Suppose $f \in \{x\} \to X$. Thus f is a function, $\mathbf{domain}\, f = \{x\}$ and $\mathbf{range}\, f \subseteq X$. Consequently, f is a relation including a pair of the form (x, y), for some $y \in X$. Because $\mathbf{domain}\, f = \{x\}$, there are no elements of f whose left sides are not x. And because f is a function, there are no other pairs in f whose left sides are x. Thus $f = \{(x, y)\}$, so that $f \in \{\, \{(x, y)\} \mid y \in X \,\}$.

Suppose $f \in \{\, \{(x, y)\} \mid y \in X \,\}$. Thus $f = \{(x, y)\}$, for some $y \in X$. Hence f is a function, $\mathbf{domain}\, f = \{x\}$ and $\mathbf{range}\, f = \{y\} \subseteq X$. Thus $f \in \{x\} \to X$.

(d) To show that $X \to \{x\} = \{\{\, (y, x) \mid y \in X \,\}\}$, we show that each side is a subset of the other.

Suppose $f \in X \to \{x\}$. Thus f is a function, $\mathbf{domain}\, f = X$ and $\mathbf{range}\, f \subseteq \{x\}$. To show that $f = \{\, (y, x) \mid y \in X \,\}$, we show that each side is a subset of the other.

- Suppose $p \in f$. From our assumptions, we know that $p = (y, x)$ for some $y \in X$. Thus $p \in \{\, (y, x) \mid y \in X \,\}$.
- Suppose $p \in \{\, (y, x) \mid y \in X \,\}$, so that $p = (y, x)$ for some $y \in X$. Because f is a function and $y \in X = \mathbf{domain}\, f$, we have that $(y, x') \in f$ for some x'. But $\mathbf{range}\, f \subseteq \{x\}$, and thus $x' = x$. Hence $p = (y, x) = (y, x') \in f$.

Because $f = \{ (y, x) \mid y \in X \}$, we can conclude that $f \in \{\{ (y, x) \mid y \in X \}\}$.

Suppose $f \in \{\{ (y, x) \mid y \in X \}\}$, so that $f = \{ (y, x) \mid y \in X \}$. Thus f is a relation, $\textbf{domain}\, f \subseteq X$ and $\textbf{range}\, f \subseteq \{x\}$. Furthermore, for all $y \in X$, $(y, x) \in f$, so that $y \in \textbf{domain}\, f$. Thus $\textbf{domain}\, f = X$. Because $\textbf{range}\, f \subseteq \{x\}$, f must be a function. Summarizing, we have that f is a function, $\textbf{domain}\, f = X$ and $\textbf{range}\, f \subseteq \{x\}$, showing that $f \in X \to \{x\}$.

Problem 2

We proceed by mathematical induction.

(Basis Step) We must show that, if $0 \geq 4$, then $2^0 < 0!$. Suppose $0 \geq 4$. But this is a contradiction, and thus we can conclude that $2^0 < 0!$.

(Inductive Step) Suppose $n \in \mathbb{N}$, and assume the inductive hypothesis:

$$\text{if } n \geq 4, \text{ then } 2^n < n!.$$

We must show that,

$$\text{if } n + 1 \geq 4, \text{ then } 2^{n+1} < (n + 1)!.$$

Suppose $n + 1 \geq 4$. We must show that $2^{n+1} < (n + 1)!$. Since $n + 1 \geq 4$, we have that $n \geq 3$. There are two cases to consider.

- Suppose $n = 3$. Then

$$2^{n+1} = 2^{3+1} = 2^4 = 16 < 24 = 4! = (3 + 1)! = (n + 1)!.$$

- Suppose $n \geq 4$. By the inductive hypothesis, we have that $2^n < n!$. Furthermore, $2 < n + 1$, so that

$$2^{n+1} = 2 * 2^n < 2 * n! < (n + 1) * n! = (n + 1)!.$$

Problem 3

We proceed by strong induction. Suppose $n \in \mathbb{N}$, and assume the inductive hypothesis: for all $m \in \mathbb{N}$, if $m < n$, then

$$\text{if } m \geq 1, \text{ then there are } i, j \in \mathbb{N} \text{ such that } m = 2^i(2j + 1).$$

We must show that

$$\text{if } n \geq 1, \text{ then there are } i, j \in \mathbb{N} \text{ such that } n = 2^i(2j + 1).$$

Suppose $n \geq 1$. We must show that there are $i, j \in \mathbb{N}$ such that $n = 2^i(2j + 1)$. There are two cases to consider.

- Suppose n is odd. Then $n = 2j + 1$ for some $j \in \mathbb{N}$. Hence $n = 1(2j + 1) = 2^0(2j + 1)$ and $0, j \in \mathbb{N}$.

- Suppose n is even. Then $n \geq 2$, so there is an $m \in \mathbb{N}$ such that $n = 2m$ and $1 \leq m < n$. Because $m \in \mathbb{N}$ and $m < n$, the inductive hypothesis tells us that

  $$\text{if } m \geq 1, \text{ then there are } i, j \in \mathbb{N} \text{ such that } m = 2^i(2j + 1).$$

  But $m \geq 1$, and thus there are $i, j \in \mathbb{N}$ such that $m = 2^i(2j + 1)$. Hence $n = 2m = 2(2^i(2j + 1)) = 2^{i+1}(2j + 1)$ and $i + 1, j \in \mathbb{N}$.

Problem 4

We proceed by well-founded induction on R. Suppose $n \in \mathbb{Z}$, and assume the inductive hypothesis: for all $m \in \mathbb{Z}$, if $m \mathrel{R} n$, then

$$\text{there is an } l \in \mathbb{N} \text{ such that } f^l(m) = 0.$$

We must show that

$$\text{there is an } l \in \mathbb{N} \text{ such that } f^l(n) = 0.$$

There are three cases to consider.

- Suppose $n = 0$. Then $f^0(n) = n = 0$ and $0 \in \mathbb{N}$.

- Suppose $n \geq 1$. Because $1 \leq n$, we have that $|1 - n| = n - 1$. Since $n \geq 0$, it follows that $|1 - n| = n - 1 < n = |n|$, and thus $1 - n \mathrel{R} n$. Because $1 - n \in \mathbb{Z}$ and $1 - n \mathrel{R} n$, the inductive hypothesis tells us that there is an $l \in \mathbb{N}$ such that $f^l(1 - n) = 0$. Thus $f^{1+l}(n) = f^l(f^1(n)) = f^l(f\,n) = f^l(1 - n) = 0$ and $1 + l \in \mathbb{N}$.

- Suppose $n \leq -1$. Thus $1 \leq -n$, so that $|-n - 1| = -n - 1$. Since $n \leq -1$, it follows that $|-n - 1| = -n - 1 < -n = |n|$, and thus $-n - 1 \mathrel{R} n$. Because $-n - 1 \in \mathbb{Z}$ and $-n - 1 \mathrel{R} n$, the inductive hypothesis tells us that there is an $l \in \mathbb{N}$ such that $f^l(-n - 1) = 0$. Thus $f^{1+l}(n) = f^l(f^1(n)) = f^l(f\,n) = f^l(-n - 1) = 0$ and $1 + l \in \mathbb{N}$.