CS 516—Software Foundations via Formal Languages—Spring 2022

Problem Set 1

Due by 5pm on Friday, February 4
Submission via Gradescope

Problem 1 (25 points)

Suppose X is a set, and $x \in X$. What are the elements of $\emptyset \to X$, $X \to \emptyset$, $\{x\} \to X$ and $X \to \{x\}$? Prove that your answers are correct.

Problem 2 (25 points)

Use mathematical induction (Theorem 1.2.1) to prove that, for all $n \in \mathbb{N}$, if $n \geq 4$, then $2^n < n!$. Here $n!$ is the factorial of n, defined recursively by:

(1) $0! = 1$;

(2) $(n+1)! = (n+1) * n!$, for all $n \in \mathbb{N}$.

Problem 3 (25 points)

Use strong induction (Theorem 1.2.4) to prove that, for all $n \in \mathbb{N}$, if $n \geq 1$, then there are $i, j \in \mathbb{N}$ such that $n = 2^i(2j+1)$.

Problem 4 (25 points)

Define $f \in \mathbb{Z} \to \mathbb{Z}$ by:

$$f\,n = \begin{cases} 0 & \text{if } n = 0, \\ 1 - n & \text{if } n \geq 1, \\ -n - 1 & \text{if } n \leq -1. \end{cases}$$

Given $n \in \mathbb{Z}$, define the l-th *iteration of* f *on* n ($f^l(n) \in \mathbb{Z}$), for $l \in \mathbb{N}$, by recursion:

$$f^0(n) = n,$$
$$f^{l+1}(n) = f(f^l(n)).$$

Then we have that:

(1) For all $n \in \mathbb{Z}$, $f^1(n) = f\,n$.

(2) For all $n \in \mathbb{Z}$ and $l, m \in \mathbb{N}$, $f^{l+m}(n) = f^m(f^l(n))$.

Let R be the relation on $\mathbb{Z}$ defined by: for all $n, m \in \mathbb{Z}$, $n \mathrel{R} m$ iff $|n| < |m|$. Here $|n|$ is the absolute value of n: n, if $n \geq 0$; and $-n$, if $n \leq -1$. We have the following properties of absolute value, where $n, m \in \mathbb{Z}$:

(1) if $n \leq m$, then $|n - m| = m - n$;

(2) if $m \leq n$, then $|n - m| = n - m$.

Because $|\cdot| \in \mathbb{Z} \to \mathbb{N}$ and $<$ is well-founded on $\mathbb{N}$, Proposition 1.2.10 tells us that R is well-founded on $\mathbb{Z}$.

Use well-founded induction on R (Theorem 1.2.8) to show that, for all $n \in \mathbb{Z}$, there is an $l \in \mathbb{N}$ such that $f^l(n) = 0$.