Chapter 1: Mathematical Background

This chapter consists of the material on set theory, induction, inductive definitions and recursion that will be required in later chapters.

1.1: Basic Set Theory

In this section, we will cover the material on logic, sets, relations, functions and data structures that will be needed in what follows.

Much of this material should be at least partly familiar.

The book starts with a review of classical logic.

Describing Sets by Listing Their Elements; Sets of Numbers

We write $\emptyset$ for the *empty set*—the set with no elements. Finite sets can be described by listing their elements inside set braces: $\{x_1, \ldots, x_n\}$.

Describing Sets by Listing Their Elements; Sets of Numbers

We write $\emptyset$ for the *empty set*—the set with no elements. Finite sets can be described by listing their elements inside set braces: $\{x_1, \ldots, x_n\}$.

We write:

- $\mathbb{N}$ for the set $\{\quad 1, \ldots\}$ of all natural numbers;
- $\mathbb{Z}$ for the set $\{\ldots, -1, 0, 1, \ldots\}$ of all integers;
- $\mathbb{R}$ for the set of all real numbers.

Describing Sets by Listing Their Elements; Sets of Numbers

We write $\emptyset$ for the *empty set*—the set with no elements. Finite sets can be described by listing their elements inside set braces: $\{x_1, \ldots, x_n\}$.

We write:

- $\mathbb{N}$ for the set $\{0, 1, \ldots\}$ of all natural numbers;
- $\mathbb{Z}$ for the set $\{\ldots, -1, 0, 1, \ldots\}$ of all integers;
- $\mathbb{R}$ for the set of all real numbers.

Relationships between Sets

Sets A and B are equal ($A = B$) iff (if and only if) they have the same elements, i.e., for all x, $x \in A$ iff $x \in B$.

Suppose A and B are sets. We say that:

- A is a *subset* of B ($A \subseteq B$) iff, for all $x \in A$, $x \in B$;
- A is a *proper subset* of B ($A \subsetneq B$) iff $A \subseteq B$ but $A \neq B$.

For example, $\emptyset \quad \mathbb{N}$, $\mathbb{N} \quad \mathbb{N}$ and $\mathbb{N} \quad \mathbb{Z}$.

Of course, $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

We also have the notions of superset ($A \supseteq B$) and proper superset ($A \supsetneq B$).

Relationships between Sets

Sets A and B are equal ($A = B$) iff (if and only if) they have the same elements, i.e., for all x, $x \in A$ iff $x \in B$.

Suppose A and B are sets. We say that:

- A is a *subset* of B ($A \subseteq B$) iff, for all $x \in A$, $x \in B$;
- A is a *proper subset* of B ($A \subsetneq B$) iff $A \subseteq B$ but $A \neq B$.

For example, $\emptyset \subsetneq \mathbb{N}$, $\mathbb{N} \quad \mathbb{N}$ and $\mathbb{N} \quad \mathbb{Z}$.

Of course, $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

We also have the notions of superset ($A \supseteq B$) and proper superset ($A \supsetneq B$).

Relationships between Sets

Sets A and B are equal ($A = B$) iff (if and only if) they have the same elements, i.e., for all x, $x \in A$ iff $x \in B$.

Suppose A and B are sets. We say that:

- A is a *subset* of B ($A \subseteq B$) iff, for all $x \in A$, $x \in B$;
- A is a *proper subset* of B ($A \subsetneq B$) iff $A \subseteq B$ but $A \neq B$.

For example, $\emptyset \subsetneq \mathbb{N}$, $\mathbb{N} \subseteq \mathbb{N}$ and $\mathbb{N} \quad \mathbb{Z}$.

Of course, $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

We also have the notions of superset ($A \supseteq B$) and proper superset ($A \supsetneq B$).

Relationships between Sets

Sets A and B are equal ($A = B$) iff (if and only if) they have the same elements, i.e., for all x, $x \in A$ iff $x \in B$.

Suppose A and B are sets. We say that:

- A is a *subset* of B ($A \subseteq B$) iff, for all $x \in A$, $x \in B$;
- A is a *proper subset* of B ($A \subsetneq B$) iff $A \subseteq B$ but $A \neq B$.

For example, $\emptyset \subsetneq \mathbb{N}$, $\mathbb{N} \subseteq \mathbb{N}$ and $\mathbb{N} \subsetneq \mathbb{Z}$.

Of course, $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

We also have the notions of superset ($A \supseteq B$) and proper superset ($A \supsetneq B$).

Set Formation

We will make extensive use of the $\{\cdots \mid \cdots\}$ notation for forming sets. Let's consider two representative examples of its use.

Let

$$A = \{\, n \mid n \in \mathbb{N} \text{ and } n^2 \geq 20 \,\} = \{\, n \in \mathbb{N} \mid n^2 \geq 20 \,\}.$$

Then, for all n,

$$n \in A \quad \text{iff} \quad n \in \mathbb{N} \text{ and } n^2 \geq 20.$$

Is $5 \in A$?

Is $5.5 \in A$?

Is $4 \in A$?

Set Formation

We will make extensive use of the $\{ \cdots \mid \cdots \}$ notation for forming sets. Let's consider two representative examples of its use.

Let

$$A = \{ n \mid n \in \mathbb{N} \text{ and } n^2 \geq 20 \} = \{ n \in \mathbb{N} \mid n^2 \geq 20 \}.$$

Then, for all n,

$$n \in A \quad \text{iff} \quad n \in \mathbb{N} \text{ and } n^2 \geq 20.$$

Is $5 \in A$? Yes—$5 \in \mathbb{N}$ and $5^2 \geq 20$.

Is $5.5 \in A$?

Is $4 \in A$?

Set Formation

We will make extensive use of the $\{ \cdots \mid \cdots \}$ notation for forming sets. Let's consider two representative examples of its use.

Let

$$A = \{ n \mid n \in \mathbb{N} \text{ and } n^2 \geq 20 \} = \{ n \in \mathbb{N} \mid n^2 \geq 20 \}.$$

Then, for all n,

$$n \in A \quad \text{iff} \quad n \in \mathbb{N} \text{ and } n^2 \geq 20.$$

Is $5 \in A$? Yes—$5 \in \mathbb{N}$ and $5^2 \geq 20$.

Is $5.5 \in A$? No—$5.5 \notin \mathbb{N}$.

Is $4 \in A$?

Set Formation

We will make extensive use of the $\{ \cdots \mid \cdots \}$ notation for forming sets. Let's consider two representative examples of its use.

Let

$$A = \{ n \mid n \in \mathbb{N} \text{ and } n^2 \geq 20 \} = \{ n \in \mathbb{N} \mid n^2 \geq 20 \}.$$

Then, for all n,

$$n \in A \quad \text{iff} \quad n \in \mathbb{N} \text{ and } n^2 \geq 20.$$

Is $5 \in A$? Yes—$5 \in \mathbb{N}$ and $5^2 \geq 20$.

Is $5.5 \in A$? No—$5.5 \notin \mathbb{N}$.

Is $4 \in A$? No—$4^2 \not\geq 20$.

Set Formation (Cont.)

Let

$$B = \{ n^3 + m^2 \mid n, m \in \mathbb{N} \text{ and } n, m \geq 1 \}.$$

Then, for all l,

Set Formation (Cont.)

Let

$$B = \{\, n^3 + m^2 \mid n, m \in \mathbb{N} \text{ and } n, m \geq 1 \,\}.$$

Then, for all l,

$$\begin{aligned} l \in B \quad &\text{iff} \quad l = n^3 + m^2, \text{ for some } n, m \text{ such that } n, m \in \mathbb{N} \text{ and } n, m \geq 1 \\ &\text{iff} \quad l = n^3 + m^2, \text{ for some } n, m \in \mathbb{N} \text{ such that } n, m \geq 1. \end{aligned}$$

Is $9 \in B$?

Set Formation (Cont.)

Let

$$B = \{ n^3 + m^2 \mid n, m \in \mathbb{N} \text{ and } n, m \geq 1 \}.$$

Then, for all l,

$$\begin{aligned} l \in B \quad \text{iff} \quad & l = n^3 + m^2, \text{ for some } n, m \text{ such that } n, m \in \mathbb{N} \text{ and } n, m \geq 1 \\ \text{iff} \quad & l = n^3 + m^2, \text{ for some } n, m \in \mathbb{N} \text{ such that } n, m \geq 1. \end{aligned}$$

Is $9 \in B$? To answer "yes", we must show

$$9 = n^3 + m^2 \text{ and } n, m \in \mathbb{N} \text{ and } n, m \geq 1,$$

for some values of n, m.

Set Formation (Cont.)

Let

$$B = \{ n^3 + m^2 \mid n, m \in \mathbb{N} \text{ and } n, m \geq 1 \}.$$

Then, for all l,

$$\begin{aligned} l \in B \quad \text{iff} \quad & l = n^3 + m^2, \text{ for some } n, m \text{ such that } n, m \in \mathbb{N} \text{ and } n, m \geq 1 \\ \text{iff} \quad & l = n^3 + m^2, \text{ for some } n, m \in \mathbb{N} \text{ such that } n, m \geq 1. \end{aligned}$$

Is $9 \in B$? To answer "yes", we must show

$$9 = n^3 + m^2 \text{ and } n, m \in \mathbb{N} \text{ and } n, m \geq 1,$$

for some values of n, m. Yes—$9 = 2^3 + 1^2$ and $2, 1 \in \mathbb{N}$ and $2, 1 \geq 1$.

Set Formation (Cont.)

Given $n, m \in \mathbb{Z}$, we write $[n : m]$ for $\{ l \in \mathbb{Z} \mid l \geq n \text{ and } l \leq m \}$.

Thus $[n : m]$ is all of the integers that are at least n and no more than m.

For example, $[-2 : 1]$ is $\{-2, -1, 0, 1\}$ and $[3 : 2]$ is

Set Formation (Cont.)

Given $n, m \in \mathbb{Z}$, we write $[n : m]$ for $\{ l \in \mathbb{Z} \mid l \geq n \text{ and } l \leq m \}$.

Thus $[n : m]$ is all of the integers that are at least n and no more than m.

For example, $[-2 : 1]$ is $\{-2, -1, 0, 1\}$ and $[3 : 2]$ is $\emptyset$.

Operations on Sets

Recall the following operations on sets A and B:

$$
\begin{aligned}
A \cup B &= \{\, x \mid x \in A \text{ or } x \in B \,\} && \text{(union)}\\
A \cap B &= \{\, x \mid x \in A \text{ and } x \in B \,\} && \text{(intersection)}\\
A - B &= \{\, x \in A \mid x \notin B \,\} && \text{(difference)}\\
A \times B &= \{\, (x, y) \mid x \in A \text{ and } y \in B \,\} && \text{(product)}\\
\mathcal{P}\, A &= \{\, X \mid X \subseteq A \,\} && \text{(power set)}.
\end{aligned}
$$

Operations on Sets

Recall the following operations on sets A and B:

$$
\begin{aligned}
A \cup B &= \{\, x \mid x \in A \text{ or } x \in B \,\} && \text{(union)} \\
A \cap B &= \{\, x \mid x \in A \text{ and } x \in B \,\} && \text{(intersection)} \\
A - B &= \{\, x \in A \mid x \notin B \,\} && \text{(difference)} \\
A \times B &= \{\, (x, y) \mid x \in A \text{ and } y \in B \,\} && \text{(product)} \\
\mathcal{P}\, A &= \{\, X \mid X \subseteq A \,\} && \text{(power set).}
\end{aligned}
$$

$A - B$ is formed by removing the elements of B from A, if necessary. For example, $\{0, 1, 2\} - \{1, 4\} =$

Operations on Sets

Recall the following operations on sets A and B:

$$
\begin{aligned}
A \cup B &= \{\, x \mid x \in A \text{ or } x \in B \,\} && \text{(union)} \\
A \cap B &= \{\, x \mid x \in A \text{ and } x \in B \,\} && \text{(intersection)} \\
A - B &= \{\, x \in A \mid x \notin B \,\} && \text{(difference)} \\
A \times B &= \{\, (x, y) \mid x \in A \text{ and } y \in B \,\} && \text{(product)} \\
\mathcal{P}\, A &= \{\, X \mid X \subseteq A \,\} && \text{(power set).}
\end{aligned}
$$

$A - B$ is formed by removing the elements of B from A, if necessary. For example, $\{0, 1, 2\} - \{1, 4\} = \{0, 2\}$.

Operations on Sets

Recall the following operations on sets A and B:

$$\begin{aligned} A \cup B &= \{\, x \mid x \in A \text{ or } x \in B \,\} && \text{(union)} \\ A \cap B &= \{\, x \mid x \in A \text{ and } x \in B \,\} && \text{(intersection)} \\ A - B &= \{\, x \in A \mid x \notin B \,\} && \text{(difference)} \\ A \times B &= \{\, (x, y) \mid x \in A \text{ and } y \in B \,\} && \text{(product)} \\ \mathcal{P}\, A &= \{\, X \mid X \subseteq A \,\} && \text{(power set).} \end{aligned}$$

$A - B$ is formed by removing the elements of B from A, if necessary. For example, $\{0, 1, 2\} - \{1, 4\} = \{0, 2\}$. $A \times B$ consists of all ordered pairs (x, y), where x comes from A and y comes from B. For example, $\{0, 1\} \times \{1, 2\} =$

Operations on Sets

Recall the following operations on sets A and B:

$$A \cup B = \{\, x \mid x \in A \text{ or } x \in B \,\} \qquad \text{(union)}$$
$$A \cap B = \{\, x \mid x \in A \text{ and } x \in B \,\} \qquad \text{(intersection)}$$
$$A - B = \{\, x \in A \mid x \notin B \,\} \qquad \text{(difference)}$$
$$A \times B = \{\, (x, y) \mid x \in A \text{ and } y \in B \,\} \qquad \text{(product)}$$
$$\mathcal{P}\,A = \{\, X \mid X \subseteq A \,\} \qquad \text{(power set).}$$

$A - B$ is formed by removing the elements of B from A, if necessary. For example, $\{0, 1, 2\} - \{1, 4\} = \{0, 2\}$. $A \times B$ consists of all ordered pairs (x, y), where x comes from A and y comes from B. For example, $\{0, 1\} \times \{1, 2\} = \{(0, 1), (0, 2), (1, 1), (1, 2)\}$.

Operations on Sets

Recall the following operations on sets A and B:

$$
\begin{aligned}
A \cup B &= \{\, x \mid x \in A \text{ or } x \in B \,\} && \text{(union)} \\
A \cap B &= \{\, x \mid x \in A \text{ and } x \in B \,\} && \text{(intersection)} \\
A - B &= \{\, x \in A \mid x \notin B \,\} && \text{(difference)} \\
A \times B &= \{\, (x, y) \mid x \in A \text{ and } y \in B \,\} && \text{(product)} \\
\mathcal{P}\, A &= \{\, X \mid X \subseteq A \,\} && \text{(power set).}
\end{aligned}
$$

$A - B$ is formed by removing the elements of B from A, if necessary. For example, $\{0, 1, 2\} - \{1, 4\} = \{0, 2\}$. $A \times B$ consists of all ordered pairs (x, y), where x comes from A and y comes from B. For example, $\{0, 1\} \times \{1, 2\} = \{(0, 1), (0, 2), (1, 1), (1, 2)\}$. We can also write $A \times B \times C$, etc.

Operations on Sets

Recall the following operations on sets A and B:

$$
\begin{aligned}
A \cup B &= \{\, x \mid x \in A \text{ or } x \in B \,\} && \text{(union)}\\
A \cap B &= \{\, x \mid x \in A \text{ and } x \in B \,\} && \text{(intersection)}\\
A - B &= \{\, x \in A \mid x \notin B \,\} && \text{(difference)}\\
A \times B &= \{\, (x, y) \mid x \in A \text{ and } y \in B \,\} && \text{(product)}\\
\mathcal{P}\, A &= \{\, X \mid X \subseteq A \,\} && \text{(power set).}
\end{aligned}
$$

$A - B$ is formed by removing the elements of B from A, if necessary. For example, $\{0, 1, 2\} - \{1, 4\} = \{0, 2\}$. $A \times B$ consists of all ordered pairs (x, y), where x comes from A and y comes from B. For example, $\{0, 1\} \times \{1, 2\} = \{(0, 1), (0, 2), (1, 1), (1, 2)\}$. We can also write $A \times B \times C$, etc. Finally, $\mathcal{P}\, A$ consists of all of the subsets of A. For example, $\mathcal{P}\, \{0, 1\} =$

Operations on Sets

Recall the following operations on sets A and B:

$$
\begin{aligned}
A \cup B &= \{\, x \mid x \in A \text{ or } x \in B \,\} && \text{(union)} \\
A \cap B &= \{\, x \mid x \in A \text{ and } x \in B \,\} && \text{(intersection)} \\
A - B &= \{\, x \in A \mid x \notin B \,\} && \text{(difference)} \\
A \times B &= \{\, (x, y) \mid x \in A \text{ and } y \in B \,\} && \text{(product)} \\
\mathcal{P}\, A &= \{\, X \mid X \subseteq A \,\} && \text{(power set).}
\end{aligned}
$$

$A - B$ is formed by removing the elements of B from A, if necessary. For example, $\{0, 1, 2\} - \{1, 4\} = \{0, 2\}$. $A \times B$ consists of all ordered pairs (x, y), where x comes from A and y comes from B. For example, $\{0, 1\} \times \{1, 2\} = \{(0, 1), (0, 2), (1, 1), (1, 2)\}$. We can also write $A \times B \times C$, etc. Finally, $\mathcal{P}\, A$ consists of all of the subsets of A. For example, $\mathcal{P}\, \{0, 1\} = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$.

Generalized Union and Intersection

If X is a set of sets, then the *generalized union* of X ($\bigcup X$) is

$$\{\, a \mid a \in A, \text{for some } A \in X \,\}.$$

For example

$$\bigcup\{\{0,1\},\{1,2\},\{2,3\}\} =$$

$$\bigcup\emptyset =$$

If X is a *nonempty* set of sets, then the *generalized intersection* of X ($\bigcap X$) is

$$\{\, a \mid a \in A, \text{for all } A \in X \,\}.$$

For example

$$\bigcap\{\{0,1\},\{1,2\},\{2,3\}\} =$$

Generalized Union and Intersection

If X is a set of sets, then the *generalized union* of X ($\bigcup X$) is

$$\{\, a \mid a \in A, \text{for some } A \in X \,\}.$$

For example

$$\bigcup\{\{0,1\},\{1,2\},\{2,3\}\} = \{0,1,2,3\} = \{0,1\} \cup \{1,2\} \cup \{2,3\},$$
$$\bigcup\emptyset =$$

If X is a *nonempty* set of sets, then the *generalized intersection* of X ($\bigcap X$) is

$$\{\, a \mid a \in A, \text{for all } A \in X \,\}.$$

For example

$$\bigcap\{\{0,1\},\{1,2\},\{2,3\}\} =$$

Generalized Union and Intersection

If X is a set of sets, then the *generalized union* of X ($\bigcup X$) is

$$\{\, a \mid a \in A, \text{for some } A \in X \,\}.$$

For example

$$\bigcup\{\{0,1\},\{1,2\},\{2,3\}\} = \{0,1,2,3\} = \{0,1\} \cup \{1,2\} \cup \{2,3\},$$
$$\bigcup \emptyset = \emptyset.$$

If X is a *nonempty* set of sets, then the *generalized intersection* of X ($\bigcap X$) is

$$\{\, a \mid a \in A, \text{for all } A \in X \,\}.$$

For example

$$\bigcap\{\{0,1\},\{1,2\},\{2,3\}\} =$$

Generalized Union and Intersection

If X is a set of sets, then the *generalized union* of X ($\bigcup X$) is

$$\{\, a \mid a \in A, \text{for some } A \in X \,\}.$$

For example

$$\bigcup\{\{0,1\},\{1,2\},\{2,3\}\} = \{0,1,2,3\} = \{0,1\} \cup \{1,2\} \cup \{2,3\},$$

$$\bigcup \emptyset = \emptyset.$$

If X is a *nonempty* set of sets, then the *generalized intersection* of X ($\bigcap X$) is

$$\{\, a \mid a \in A, \text{for all } A \in X \,\}.$$

For example

$$\bigcap\{\{0,1\},\{1,2\},\{2,3\}\} = \emptyset = \{0,1\} \cap \{1,2\} \cap \{2,3\}.$$

Relations and Functions

A *relation* R is a set of ordered pairs.

The *domain* of a relation R ($\mathbf{domain}\, R$) is $\{\, x \mid (x, y) \in R,$ for some $y \,\}$, and the *range* of R ($\mathbf{range}\, R$) is $\{\, y \mid (x, y) \in R,$ for some $x \,\}$.

We say that R is a *relation from* a set X *to* a set Y iff $\mathbf{domain}\, R \subseteq X$ and $\mathbf{range}\, R \subseteq Y$, and that R is a *relation on* a set A iff $\mathbf{domain}\, R \cup \mathbf{range}\, R \subseteq A$.

We often write $x\, R\, y$ for $(x, y) \in R$.

Consider the relation

$$R = \{(0, 1), (1, 2), (0, 2)\}.$$

Then, $\mathbf{domain}\, R =$, $\mathbf{range}\, R =$, R is a relation from to , and R is a relation on

A *relation* R is a set of ordered pairs.

The *domain* of a relation R (**domain** R) is $\{ x \mid (x, y) \in R,$ for some $y \}$, and the *range* of R (**range** R) is $\{ y \mid (x, y) \in R,$ for some $x \}$.

We say that R is a *relation from* a set X *to* a set Y iff **domain** $R \subseteq X$ and **range** $R \subseteq Y$, and that R is a *relation on* a set A iff **domain** $R \cup$ **range** $R \subseteq A$.

We often write $x \, R \, y$ for $(x, y) \in R$.

Consider the relation

$$R = \{(0, 1), (1, 2), (0, 2)\}.$$

Then, **domain** $R = \{0, 1\}$, **range** $R =$, R is a relation from to , and R is a relation on

A *relation* R is a set of ordered pairs.

The *domain* of a relation R (**domain** R) is $\{ x \mid (x, y) \in R,$ for some $y \}$, and the *range* of R (**range** R) is $\{ y \mid (x, y) \in R,$ for some $x \}$.

We say that R is a *relation from* a set X *to* a set Y iff **domain** $R \subseteq X$ and **range** $R \subseteq Y$, and that R is a *relation on* a set A iff **domain** $R \cup$ **range** $R \subseteq A$.

We often write $x \, R \, y$ for $(x, y) \in R$.

Consider the relation

$$R = \{(0, 1), (1, 2), (0, 2)\}.$$

Then, **domain** $R = \{0, 1\}$, **range** $R = \{1, 2\}$, R is a relation from to , and R is a relation on

A *relation* R is a set of ordered pairs.

The *domain* of a relation R ($\textbf{domain}\, R$) is $\{ x \mid (x, y) \in R,$ for some $y \}$, and the *range* of R ($\textbf{range}\, R$) is $\{ y \mid (x, y) \in R,$ for some $x \}$.

We say that R is a *relation from* a set X *to* a set Y iff $\textbf{domain}\, R \subseteq X$ and $\textbf{range}\, R \subseteq Y$, and that R is a *relation on* a set A iff $\textbf{domain}\, R \cup \textbf{range}\, R \subseteq A$.

We often write $x\, R\, y$ for $(x, y) \in R$.

Consider the relation

$$R = \{(0, 1), (1, 2), (0, 2)\}.$$

Then, $\textbf{domain}\, R = \{0, 1\}$, $\textbf{range}\, R = \{1, 2\}$, R is a relation from $\{0, 1\}$ to , and R is a relation on

A *relation* R is a set of ordered pairs.

The *domain* of a relation R ($\mathbf{domain}\, R$) is $\{\, x \mid (x, y) \in R,$ for some $y \,\}$, and the *range* of R ($\mathbf{range}\, R$) is $\{\, y \mid (x, y) \in R,$ for some $x \,\}$.

We say that R is a *relation from* a set X *to* a set Y iff $\mathbf{domain}\, R \subseteq X$ and $\mathbf{range}\, R \subseteq Y$, and that R is a *relation on* a set A iff $\mathbf{domain}\, R \cup \mathbf{range}\, R \subseteq A$.

We often write $x \; R \; y$ for $(x, y) \in R$.

Consider the relation

$$R = \{(0, 1), (1, 2), (0, 2)\}.$$

Then, $\mathbf{domain}\, R = \{0, 1\}$, $\mathbf{range}\, R = \{1, 2\}$, R is a relation from $\{0, 1\}$ to $\{1, 2\}$, and R is a relation on

Relations and Functions

A *relation* R is a set of ordered pairs.

The *domain* of a relation R ($\mathbf{domain}\,R$) is $\{\, x \mid (x, y) \in R,$ for some $y \,\}$, and the *range* of R ($\mathbf{range}\,R$) is $\{\, y \mid (x, y) \in R,$ for some $x \,\}$.

We say that R is a *relation from* a set X *to* a set Y iff $\mathbf{domain}\,R \subseteq X$ and $\mathbf{range}\,R \subseteq Y$, and that R is a *relation on* a set A iff $\mathbf{domain}\,R \cup \mathbf{range}\,R \subseteq A$.

We often write $x \; R \; y$ for $(x, y) \in R$.

Consider the relation

$$R = \{(0, 1), (1, 2), (0, 2)\}.$$

Then, $\mathbf{domain}\,R = \{0, 1\}$, $\mathbf{range}\,R = \{1, 2\}$, R is a relation from $\{0, 1\}$ to $\{1, 2\}$, and R is a relation on $\{0, 1, 2\}$.

Properties of Relations

A relation R is:

- *reflexive on* a set A iff, for all $x \in A$,
- *transitive* iff, for all x, y, z, if $(x, y) \in R$ and $(y, z) \in R$, then
- *symmetric* iff, for all x, y, if $(x, y) \in R$, then
- a *function* iff, for all x, y, z, if $(x, y) \in R$ and $(x, z) \in R$, then

Is $R = \{(0, 1), (1, 2), (0, 2)\}$ reflexive on $\{0, 1, 2\}$?
Is R transitive?
Is R symmetric?
Is R a function?

.

Properties of Relations

A relation R is:

- *reflexive on* a set A iff, for all $x \in A$, $(x, x) \in R$;
- *transitive* iff, for all x, y, z, if $(x, y) \in R$ and $(y, z) \in R$, then
- *symmetric* iff, for all x, y, if $(x, y) \in R$, then
- a *function* iff, for all x, y, z, if $(x, y) \in R$ and $(x, z) \in R$, then

Is $R = \{(0, 1), (1, 2), (0, 2)\}$ reflexive on $\{0, 1, 2\}$?
Is R transitive?
Is R symmetric?
Is R a function?

.

Properties of Relations

A relation R is:

- *reflexive on* a set A iff, for all $x \in A$, $(x, x) \in R$;
- *transitive* iff, for all x, y, z, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$;
- *symmetric* iff, for all x, y, if $(x, y) \in R$, then
- a *function* iff, for all x, y, z, if $(x, y) \in R$ and $(x, z) \in R$, then

Is $R = \{(0, 1), (1, 2), (0, 2)\}$ reflexive on $\{0, 1, 2\}$?
Is R transitive?
Is R symmetric?
Is R a function?

.

Properties of Relations

A relation R is:

- *reflexive on* a set A iff, for all $x \in A$, $(x, x) \in R$;
- *transitive* iff, for all x, y, z, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$;
- *symmetric* iff, for all x, y, if $(x, y) \in R$, then $(y, x) \in R$;
- a *function* iff, for all x, y, z, if $(x, y) \in R$ and $(x, z) \in R$, then

Is $R = \{(0, 1), (1, 2), (0, 2)\}$ reflexive on $\{0, 1, 2\}$?
Is R transitive?
Is R symmetric?
Is R a function?

.

Properties of Relations

A relation R is:

- *reflexive on* a set A iff, for all $x \in A$, $(x, x) \in R$;
- *transitive* iff, for all x, y, z, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$;
- *symmetric* iff, for all x, y, if $(x, y) \in R$, then $(y, x) \in R$;
- a *function* iff, for all x, y, z, if $(x, y) \in R$ and $(x, z) \in R$, then $y = z$.

Is $R = \{(0, 1), (1, 2), (0, 2)\}$ reflexive on $\{0, 1, 2\}$?
Is R transitive?
Is R symmetric?
Is R a function?

.

Properties of Relations

A relation R is:

- *reflexive on* a set A iff, for all $x \in A$, $(x, x) \in R$;
- *transitive* iff, for all x, y, z, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$;
- *symmetric* iff, for all x, y, if $(x, y) \in R$, then $(y, x) \in R$;
- a *function* iff, for all x, y, z, if $(x, y) \in R$ and $(x, z) \in R$, then $y = z$.

Is $R = \{(0,1), (1,2), (0,2)\}$ reflexive on $\{0,1,2\}$? No—$(0,0) \notin R$.
Is R transitive?
Is R symmetric?
Is R a function?

.

Properties of Relations

A relation R is:

- *reflexive on* a set A iff, for all $x \in A$, $(x, x) \in R$;
- *transitive* iff, for all x, y, z, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$;
- *symmetric* iff, for all x, y, if $(x, y) \in R$, then $(y, x) \in R$;
- a *function* iff, for all x, y, z, if $(x, y) \in R$ and $(x, z) \in R$, then $y = z$.

Is $R = \{(0, 1), (1, 2), (0, 2)\}$ reflexive on $\{0, 1, 2\}$? No—$(0, 0) \notin R$.
Is R transitive? Yes; since $(0, 1), (1, 2) \in R$, $(0, 2) \in R$ required.
Is R symmetric?
Is R a function?

Properties of Relations

A relation R is:

- *reflexive on* a set A iff, for all $x \in A$, $(x, x) \in R$;
- *transitive* iff, for all x, y, z, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$;
- *symmetric* iff, for all x, y, if $(x, y) \in R$, then $(y, x) \in R$;
- a *function* iff, for all x, y, z, if $(x, y) \in R$ and $(x, z) \in R$, then $y = z$.

Is $R = \{(0,1), (1,2), (0,2)\}$ reflexive on $\{0,1,2\}$? No—$(0,0) \notin R$.
Is R transitive? Yes; since $(0,1), (1,2) \in R$, $(0,2) \in R$ required.
Is R symmetric? No—$(0,1) \in R$, but $(1,0) \notin R$.
Is R a function?

.

Properties of Relations

A relation R is:

- *reflexive on* a set A iff, for all $x \in A$, $(x, x) \in R$;
- *transitive* iff, for all x, y, z, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$;
- *symmetric* iff, for all x, y, if $(x, y) \in R$, then $(y, x) \in R$;
- a *function* iff, for all x, y, z, if $(x, y) \in R$ and $(x, z) \in R$, then $y = z$.

Is $R = \{(0,1), (1,2), (0,2)\}$ reflexive on $\{0,1,2\}$? No—$(0,0) \notin R$.
Is R transitive? Yes; since $(0,1), (1,2) \in R$, $(0,2) \in R$ required.
Is R symmetric? No—$(0,1) \in R$, but $(1,0) \notin R$.
Is R a function? No—$(0,1) \in R$ and $(0,2) \in R$.

Properties of Relations

A relation R is:

- *reflexive on* a set A iff, for all $x \in A$, $(x, x) \in R$;
- *transitive* iff, for all x, y, z, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$;
- *symmetric* iff, for all x, y, if $(x, y) \in R$, then $(y, x) \in R$;
- a *function* iff, for all x, y, z, if $(x, y) \in R$ and $(x, z) \in R$, then $y = z$.

Is $R = \{(0, 1), (1, 2), (0, 2)\}$ reflexive on $\{0, 1, 2\}$? No—$(0, 0) \notin R$.
Is R transitive? Yes; since $(0, 1), (1, 2) \in R$, $(0, 2) \in R$ required.
Is R symmetric? No—$(0, 1) \in R$, but $(1, 0) \notin R$.
Is R a function? No—$(0, 1) \in R$ and $(0, 2) \in R$.

The book talks about *total orderings* like $\leq$ on $\mathbb{N}$, as well as the corresponding *strict total orderings*, like $<$ on $\mathbb{N}$.

More on Functions

The relation

$$f = \{(0,1),(1,2),(2,0)\}$$

is a function.

If f is a function and $x \in \textbf{domain}\, f$, we write $f\,x$ for the *application* of f to x, i.e., the unique y such that $(x, y) \in f$.

We say that f is a *function from* a set X *to* a set Y iff f is a function, $\textbf{domain}\, f = X$ and $\textbf{range}\, f \subseteq Y$.

We write $X \to Y$ for the set of all functions from X to Y.

For the f defined above, we have that $f\,0 =$, $f\,1 =$, $f\,2 =$, f is a function from to , and
$f \in \quad \to$

More on Functions

The relation

$$f = \{(0,1), (1,2), (2,0)\}$$

is a function.

If f is a function and $x \in \textbf{domain}\, f$, we write $f\,x$ for the *application* of f to x, i.e., the unique y such that $(x, y) \in f$.

We say that f is a *function from* a set X *to* a set Y iff f is a function, $\textbf{domain}\, f = X$ and $\textbf{range}\, f \subseteq Y$.

We write $X \to Y$ for the set of all functions from X to Y.

For the f defined above, we have that $f\,0 = 1$, $f\,1 =$, $f\,2 =$, f is a function from to , and $f \in \quad \to$

More on Functions

The relation

$$f = \{(0,1),(1,2),(2,0)\}$$

is a function.

If f is a function and $x \in \textbf{domain}\, f$, we write $f\,x$ for the *application* of f to x, i.e., the unique y such that $(x, y) \in f$.

We say that f is a *function from* a set X *to* a set Y iff f is a function, $\textbf{domain}\, f = X$ and $\textbf{range}\, f \subseteq Y$.

We write $X \to Y$ for the set of all functions from X to Y.

For the f defined above, we have that $f\,0 = 1$, $f\,1 = 2$, $f\,2 =$, f is a function from to , and
$f \in \quad \to$

More on Functions

The relation

$$f = \{(0,1),(1,2),(2,0)\}$$

is a function.

If f is a function and $x \in \textbf{domain}\, f$, we write $f\, x$ for the *application* of f to x, i.e., the unique y such that $(x, y) \in f$.

We say that f is a *function from* a set X *to* a set Y iff f is a function, $\textbf{domain}\, f = X$ and $\textbf{range}\, f \subseteq Y$.

We write $X \to Y$ for the set of all functions from X to Y.

For the f defined above, we have that $f\,0 = 1$, $f\,1 = 2$, $f\,2 = 0$, f is a function from to , and
$f \in \quad \to$

More on Functions

The relation

$$f = \{(0,1),(1,2),(2,0)\}$$

is a function.

If f is a function and $x \in \textbf{domain}\, f$, we write $f\,x$ for the *application* of f to x, i.e., the unique y such that $(x,y) \in f$.

We say that f is a *function from* a set X *to* a set Y iff f is a function, $\textbf{domain}\, f = X$ and $\textbf{range}\, f \subseteq Y$.

We write $X \to Y$ for the set of all functions from X to Y.

For the f defined above, we have that $f\,0 = 1$, $f\,1 = 2$, $f\,2 = 0$, f is a function from $\{0,1,2\}$ to , and
$f \in \quad \to$

More on Functions

The relation

$$f = \{(0,1),(1,2),(2,0)\}$$

is a function.

If f is a function and $x \in \textbf{domain}\, f$, we write $f\,x$ for the *application* of f to x, i.e., the unique y such that $(x,y) \in f$.

We say that f is a *function from* a set X *to* a set Y iff f is a function, $\textbf{domain}\, f = X$ and $\textbf{range}\, f \subseteq Y$.

We write $X \to Y$ for the set of all functions from X to Y.

For the f defined above, we have that $f\,0 = 1$, $f\,1 = 2$, $f\,2 = 0$, f is a function from $\{0,1,2\}$ to $\{0,1,2\}$, and
$f \in \quad \to$

More on Functions

The relation

$$f = \{(0,1),(1,2),(2,0)\}$$

is a function.

If f is a function and $x \in \textbf{domain}\, f$, we write $f\,x$ for the *application* of f to x, i.e., the unique y such that $(x,y) \in f$.

We say that f is a *function from* a set X *to* a set Y iff f is a function, $\textbf{domain}\, f = X$ and $\textbf{range}\, f \subseteq Y$.

We write $X \to Y$ for the set of all functions from X to Y.

For the f defined above, we have that $f\,0 = 1$, $f\,1 = 2$, $f\,2 = 0$, f is a function from $\{0,1,2\}$ to $\{0,1,2\}$, and
$f \in \{0,1,2\} \to$

More on Functions

The relation

$$f = \{(0,1),(1,2),(2,0)\}$$

is a function.

If f is a function and $x \in \textbf{domain}\, f$, we write $f\, x$ for the *application* of f to x, i.e., the unique y such that $(x, y) \in f$.

We say that f is a *function from* a set X *to* a set Y iff f is a function, $\textbf{domain}\, f = X$ and $\textbf{range}\, f \subseteq Y$.

We write $X \to Y$ for the set of all functions from X to Y.

For the f defined above, we have that $f\,0 = 1$, $f\,1 = 2$, $f\,2 = 0$, f is a function from $\{0,1,2\}$ to $\{0,1,2\}$, and $f \in \{0,1,2\} \to \{0,1,2\}$.

Bijections

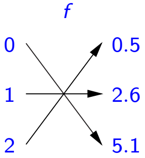
A *bijection* f *from* a set X *to* a set Y is a function from X to Y such that, for all $y \in Y$, there is a unique $x \in X$ such that $(x, y) \in f$.

For example,

$$f = \{(0, 5.1), (1, 2.6), (2, 0.5)\}$$

is a bijection from $\{0, 1, 2\}$ to $\{0.5, 2.6, 5.1\}$.

We can visualize f as a one-to-one correspondence between these sets:



Set Cardinality

We say that a set X is *equinumerous* to a set Y ($X \cong Y$) iff there is a bijection from X to Y. It's not hard to show that for all sets X, Y, Z:

- $X \cong X$;
- If $X \cong Y \cong Z$, then $X \cong Z$;
- If $X \cong Y$, then $Y \cong X$.

Finite and Infinite Sets

A set X is *finite* iff $X \cong [1 : n]$, for some $n \in \mathbb{N}$; otherwise X is *infinite*.

A set X is *countably infinite* iff $X \cong \mathbb{N}$.

A set X is *countable* iff X is either finite or countably infinite; otherwise X is *uncountable*.

Finite and Infinite Sets

A set X is *finite* iff $X \cong [1 : n]$, for some $n \in \mathbb{N}$; otherwise X is *infinite*.

A set X is *countably infinite* iff $X \cong \mathbb{N}$.

A set X is *countable* iff X is either finite or countably infinite; otherwise X is *uncountable*.

Every set X has a *size* or *cardinality* ($|X|$) and we have that, for all sets X and Y, $|X| = |Y|$ iff $X \cong Y$. The sizes of finite sets are natural numbers.

Set Size Examples

- The sets $\emptyset$ and $\{0.5, 2.6, 5.1\}$ are finite, and are thus also countable;
- The sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathcal{P}\,\mathbb{N}$ are infinite;
- The set $\mathbb{N}$ is countably infinite, and is thus countable;
- The set $\mathbb{Z}$ is
- The sets $\mathbb{R}$ and $\mathcal{P}\,\mathbb{N}$ are uncountable.

Set Size Examples

- The sets $\emptyset$ and $\{0.5, 2.6, 5.1\}$ are finite, and are thus also countable;
- The sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathcal{P}\,\mathbb{N}$ are infinite;
- The set $\mathbb{N}$ is countably infinite, and is thus countable;
- The set $\mathbb{Z}$ is countably infinite, and is thus countable, because of the existence of the following bijection:

$$\begin{array}{ccccccc} \cdots & -2 & -1 & 0 & 1 & 2 & \cdots \\ \cdots & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\ \cdots & & 2 & 0 & 1 & & \cdots \end{array}$$

- The sets $\mathbb{R}$ and $\mathcal{P}\,\mathbb{N}$ are uncountable.

Set Size Examples

- The sets $\emptyset$ and $\{0.5, 2.6, 5.1\}$ are finite, and are thus also countable;
- The sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathcal{P}\,\mathbb{N}$ are infinite;
- The set $\mathbb{N}$ is countably infinite, and is thus countable;
- The set $\mathbb{Z}$ is countably infinite, and is thus countable, because of the existence of the following bijection:

$\cdots$	-2	-1	0	1	2	$\cdots$
$\cdots$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\cdots$
$\cdots$		2	0	1	3	$\cdots$

- The sets $\mathbb{R}$ and $\mathcal{P}\,\mathbb{N}$ are uncountable.

Set Size Examples

- The sets $\emptyset$ and $\{0.5, 2.6, 5.1\}$ are finite, and are thus also countable;
- The sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathcal{P}\,\mathbb{N}$ are infinite;
- The set $\mathbb{N}$ is countably infinite, and is thus countable;
- The set $\mathbb{Z}$ is countably infinite, and is thus countable, because of the existence of the following bijection:

$$\begin{array}{ccccccc} \cdots & -2 & -1 & 0 & 1 & 2 & \cdots \\ \cdots & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\ \cdots & 4 & 2 & 0 & 1 & 3 & \cdots \end{array}$$

- The sets $\mathbb{R}$ and $\mathcal{P}\,\mathbb{N}$ are uncountable.

Data Structures: Booleans

$\mathbf{Bool} = \{\mathbf{true}, \mathbf{false}\}$.

We have the usual negation (**not**), conjunction (**and**) and disjunction (**or**) operations on booleans.

Options

$\textbf{Option}\,X = \{\textbf{none}\} \cup \{\, \textbf{some}\,x \mid x \in X \,\}$.

For example, $\textbf{Option Bool} = \{\textbf{none}, \textbf{some true}, \textbf{some false}\}$.

E.g., we could define a function $f \in \mathbb{N} \times \mathbb{N} \to \textbf{Option Bool}$ by:

$$f(n, m) = \begin{cases} \textbf{none}, & \text{if } m = 0, \\ \textbf{some true} & \text{if } m \neq 0 \text{ and } n = ml \text{ for some } l \in \mathbb{N}, \\ \textbf{some false} & \text{if } m \neq 0 \text{ and } n \neq ml \text{ for all } l \in \mathbb{N}. \end{cases}$$

Lists

A *list* is a function with domain $[1:n]$, for some $n \in \mathbb{N}$.

For example $\emptyset$ is a list, as it is a function with domain

And $\{(1,3),(2,5),(3,7)\}$ is a list, as it is a function with domain

We abbreviate a list $\{(1,x_1),(2,x_2),\ldots,(n,x_n)\}$ to $[x_1,x_2,\ldots,x_n]$.
Thus $\emptyset$ and $\{(1,3),(2,5),(3,7)\}$ are abbreviated to $[\,]$ and $[3,5,7]$.

$|\cdot|$ doubles as list

$f \mathbin{@} g$ is list concatenation. E.g., $[2,3,4] \mathbin{@} [5,6] = [2,3,4,5,6]$.

Concatenation is associative $(f \mathbin{@} g) \mathbin{@} h = f \mathbin{@} (g \mathbin{@} h)$ and has $[\,]$ as its identity ($[\,] \mathbin{@} f = f = f \mathbin{@} [\,]$).

List X is all X-*lists*, i.e., all lists whose ranges are subsets of X, i.e., whose elements come from X.

Lists

A *list* is a function with domain $[1 : n]$, for some $n \in \mathbb{N}$.

For example $\emptyset$ is a list, as it is a function with domain $\emptyset =$

And $\{(1,3),(2,5),(3,7)\}$ is a list, as it is a function with domain

We abbreviate a list $\{(1,x_1),(2,x_2),\ldots,(n,x_n)\}$ to $[x_1,x_2,\ldots,x_n]$. Thus $\emptyset$ and $\{(1,3),(2,5),(3,7)\}$ are abbreviated to $[\,]$ and $[3,5,7]$.

$|\cdot|$ doubles as list

$f \mathbin{@} g$ is list concatenation. E.g., $[2,3,4] \mathbin{@} [5,6] = [2,3,4,5,6]$.

Concatenation is associative $(f \mathbin{@} g) \mathbin{@} h = f \mathbin{@} (g \mathbin{@} h)$ and has $[\,]$ as its identity ($[\,] \mathbin{@} f = f = f \mathbin{@} [\,]$).

List X is all X-*lists*, i.e., all lists whose ranges are subsets of X, i.e., whose elements come from X.

Lists

A *list* is a function with domain $[1 : n]$, for some $n \in \mathbb{N}$.

For example $\emptyset$ is a list, as it is a function with domain $\emptyset = [1 : 0]$.

And $\{(1, 3), (2, 5), (3, 7)\}$ is a list, as it is a function with domain

We abbreviate a list $\{(1, x_1), (2, x_2), \ldots, (n, x_n)\}$ to $[x_1, x_2, \ldots, x_n]$.
Thus $\emptyset$ and $\{(1, 3), (2, 5), (3, 7)\}$ are abbreviated to $[\,]$ and $[3, 5, 7]$.

$|\cdot|$ doubles as list

$f \mathbin{@} g$ is list concatenation. E.g., $[2, 3, 4] \mathbin{@} [5, 6] = [2, 3, 4, 5, 6]$.

Concatenation is associative $(f \mathbin{@} g) \mathbin{@} h = f \mathbin{@} (g \mathbin{@} h)$ and has $[\,]$ as its identity ($[\,] \mathbin{@} f = f = f \mathbin{@} [\,]$).

List X is all X-*lists*, i.e., all lists whose ranges are subsets of X, i.e., whose elements come from X.

Lists

A *list* is a function with domain $[1:n]$, for some $n \in \mathbb{N}$.

For example $\emptyset$ is a list, as it is a function with domain $\emptyset = [1:0]$.

And $\{(1,3),(2,5),(3,7)\}$ is a list, as it is a function with domain $[1:3]$.

We abbreviate a list $\{(1,x_1),(2,x_2),\ldots,(n,x_n)\}$ to $[x_1,x_2,\ldots,x_n]$.
Thus $\emptyset$ and $\{(1,3),(2,5),(3,7)\}$ are abbreviated to $[\,]$ and $[3,5,7]$.

$|\cdot|$ doubles as list

$f \mathbin{@} g$ is list concatenation. E.g., $[2,3,4] \mathbin{@} [5,6] = [2,3,4,5,6]$.

Concatenation is associative $(f \mathbin{@} g) \mathbin{@} h = f \mathbin{@} (g \mathbin{@} h)$ and has $[\,]$ as its identity ($[\,] \mathbin{@} f = f = f \mathbin{@} [\,]$).

List X is all X-*lists*, i.e., all lists whose ranges are subsets of X, i.e., whose elements come from X.

Lists

A *list* is a function with domain $[1 : n]$, for some $n \in \mathbb{N}$.

For example $\emptyset$ is a list, as it is a function with domain $\emptyset = [1 : 0]$.

And $\{(1, 3), (2, 5), (3, 7)\}$ is a list, as it is a function with domain $[1 : 3]$.

We abbreviate a list $\{(1, x_1), (2, x_2), \ldots, (n, x_n)\}$ to $[x_1, x_2, \ldots, x_n]$. Thus $\emptyset$ and $\{(1, 3), (2, 5), (3, 7)\}$ are abbreviated to $[\,]$ and $[3, 5, 7]$.

$|\cdot|$ doubles as list *length*.

$f \mathbin{@} g$ is list concatenation. E.g., $[2, 3, 4] \mathbin{@} [5, 6] = [2, 3, 4, 5, 6]$.

Concatenation is associative $(f \mathbin{@} g) \mathbin{@} h = f \mathbin{@} (g \mathbin{@} h)$ and has $[\,]$ as its identity ($[\,] \mathbin{@} f = f = f \mathbin{@} [\,]$).

List X is all X-*lists*, i.e., all lists whose ranges are subsets of X, i.e., whose elements come from X.