1.2: Induction

In the section, we consider several induction principles, i.e., methods for proving that every element x of some set A has some property $P(x)$.

Theorem 1.2.1 (Principle of Mathematical Induction)
Suppose $P(n)$ is a property of a natural number n.
If

(basis step)

$$P(0) \text{ and}$$

(inductive step)

$$\text{for all } n \in \mathbb{N}, \text{ if } (\dagger)\ P(n), \text{ then } P(n+1),$$

then,

$$\text{for all } n \in \mathbb{N},\ P(n).$$

We refer to the formula (†) as the *inductive hypothesis*.

Principle of Strong Induction

Theorem 1.2.4 (Principle of Strong Induction)
Suppose $P(n)$ is a property of a natural number n.
If

> *for all $n \in \mathbb{N}$,*
> *if (†) for all $m \in \mathbb{N}$, if $m < n$, then $P(m)$,*
> *then $P(n)$,*

then

$$\text{for all } n \in \mathbb{N},\ P(n).$$

We refer to the formula (†) as the *inductive hypothesis*.

Proof. Follows by mathematical induction, but using a property $Q(n)$ derived from $P(n)$. See the book. □

Example Proof Using Strong Induction

Proposition 1.2.5

Every nonempty set of natural numbers has a least element.

Proof. Let X be a nonempty set of natural numbers.

We begin by using strong induction to show that, for all $n \in \mathbb{N}$,

$$\text{if } n \in X, \text{ then } X \text{ has a least element.}$$

Suppose $n \in \mathbb{N}$, and assume the inductive hypothesis: for all $m \in \mathbb{N}$, if $m < n$, then

$$\text{if } m \in X, \text{ then } X \text{ has a least element.}$$

We must show that

$$\text{if } n \in X, \text{ then } X \text{ has a least element.}$$

Example Proof (Cont.)

Proof (cont.). Suppose $n \in X$. It remains to show that X has a least element. If n is less-than-or-equal-to every element of X, then we are done. Otherwise, there is an $m \in X$ such that $m < n$. By the inductive hypothesis, we have that

if $m \in X$, then X has a least element.

But $m \in X$, and thus X has a least element. This completes our strong induction.

Example Proof (Cont.)

Proof (cont.). Now we use the result of our strong induction to prove that X has a least element. Since X is a nonempty subset of $\mathbb{N}$, there is an $n \in \mathbb{N}$ such that $n \in X$. By the result of our induction, we can conclude that

$$\text{if } n \in X, \text{ then } X \text{ has a least element.}$$

But $n \in X$, and thus X has a least element. $\square$

Well-founded Induction

We can also do induction over a well-founded relation.

A relation R on a set A is *well-founded* iff every nonempty subset X of A has an R-minimal element, where an element $x \in X$ is *R-minimal in X* iff there is no $y \in X$ such that $y \mathrel{R} x$.

Given $x, y \in A$, we say that *y is a predecessor of x in R* iff $y \mathrel{R} x$. Thus $x \in X$ is R-minimal in X iff none of x's predecessors in R (there may be none) are in X.

For example, in Proposition 1.2.5, we proved that the strict total ordering $<$ on $\mathbb{N}$ is well-founded.

On the other hand, the strict total ordering $<$ on $\mathbb{Z}$ is *not* well-founded, as $\mathbb{Z}$ itself lacks a $<$-minimal element.

Well-founded Induction (Cont.)

Here's another negative example, showing that even if the underlying set is finite, the relation need not be well-founded.

Let $A = \{0, 1\}$, and $R = \{(0, 1), (1, 0)\}$. Then 0 is the only predecessor of 1 in R, and 1 is the only predecessor of 0 in R.

Of the nonempty subsets of A, we have that $\{0\}$ and $\{1\}$ have R-minimal elements. But consider A itself. Then 0 is not R-minimal in A, because $1 \in A$ and $1 \; R \; 0$. And 1 is not R-minimal in A, because $0 \in A$ and $0 \; R \; 1$. Hence R is not well-founded.

Principle of Well-founded Induction

Theorem 1.2.8 (Principle of Well-founded Induction)
Suppose A is a set, R is a well-founded relation on A, and $P(x)$ is a property of an element $x \in A$.
If

for all $x \in A$,
if (†) *for all* $y \in A$, *if* $y \; R \; x$, *then* $P(y)$,
then $P(x)$,

then

$$\text{for all } x \in A,\ P(x).$$

We refer to the formula (†) as the *inductive hypothesis*.

When $A = \mathbb{N}$ and $R = <$, this is the same as the principle of strong induction.

Proof of Well-founded Induction

Proof. Suppose A is a set, R is a well-founded relation on A, $P(x)$ is a property of an element $x \in A$, and

(‡) for all $x \in A$,
if for all $y \in A$, if $y \mathrel{R} x$, then $P(y)$,
then $P(x)$.

We must show that, for all $x \in A$, $P(x)$.

Suppose, toward a contradiction, that it is not the case that, for all $x \in A$, $P(x)$. Hence there is an $x \in A$ such that $P(x)$ is false. Let $X = \{\, x \in A \mid P(x) \text{ is false} \,\}$. Thus $x \in X$, showing that X is non-empty. Because R is well-founded on A, it follows that there is a $z \in X$ that is R-minimal in X, i.e., such that there is no $y \in X$ such that $y \mathrel{R} z$.

Proof of Well-founded Induction (Cont.)

Proof (cont.). By (‡), we have that

if for all $y \in A$, if $y \, R \, z$, then $P(y)$,
then $P(z)$.

Because $z \in X$, we have that $P(z)$ is false. Thus, to obtain a contradiction, it will suffice to show that

for all $y \in A$, if $y \, R \, z$, then $P(y)$.

Suppose $y \in A$, and $y \, R \, z$. We must show that $P(y)$. Because z is R-minimal in X, it follows that $y \notin X$. Thus $P(y)$. □

Well-founded Induction on Predecessor Relation

Let the predecessor relation $\mathbf{pred}_{\mathbb{N}}$ on $\mathbb{N}$ be $\{\,(n, n+1) \mid n \in \mathbb{N}\,\}$.

Then $\mathbf{pred}_{\mathbb{N}}$ is well-founded on $\mathbb{N}$, because $\mathbf{pred}_{\mathbb{N}} \subseteq <$ and $<$ is well-founded on $\mathbb{N}$ (see Proposition 1.2.9 in the book).

0 has no predecessors in $\mathbf{pred}_{\mathbb{N}}$, and, for all $n \in \mathbb{N}$, n is the only predecessor of $n+1$ in $\mathbf{pred}_{\mathbb{N}}$. Consequently, if a zero/non-zero case analysis is used, a proof by well-founded induction on $\mathbf{pred}_{\mathbb{N}}$ will look like a proof by mathematical induction.

Well-founded Induction on Integers via Absolute Value

Let R be the relation on $\mathbb{Z}$ such that, for all $n, m \in \mathbb{Z}$, $n \; R \; m$ iff $|n| < |m|$.

Since $|\cdot| \in \mathbb{Z} \to \mathbb{N}$ and $<$ is well-founded on $\mathbb{N}$, Proposition 1.2.10 from the book tells us that R is well-founded on $\mathbb{Z}$.

If we do a well-founded induction on R, when proving $P(n)$, for $n \in \mathbb{Z}$, we can make use of $P(m)$ for any $m \in \mathbb{Z}$ whose absolute value is strictly less than the absolute value of n.

E.g., when proving $P(-10)$, we could make use of $P(5)$ or $P(-9)$.