# *1.2: Induction*

In the section, we consider several induction principles, i.e., methods for proving that every element $x$ of some set $A$ has some property $P(x)$.

# Principle of Mathematical Induction

**Theorem 1.2.1 (Principle of Mathematical Induction)**
*Suppose $P(n)$ is a property of a natural number $n$.*
*If*

*(basis step)*

*and*

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*for all $n \in \mathbb{N}$, if then*

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$$\text{for all } n \in \mathbb{N}, \text{ if } (\dagger)\ P(n), \text{ then } P(n+1),$$

*then,*

$$\text{for all } n \in \mathbb{N},\ P(n).$$

We refer to the formula (†) as the *inductive hypothesis*.

**Theorem 1.2.4 (Principle of Strong Induction)**
*Suppose $P(n)$ is a property of a natural number $n$.*
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*for all $n \in \mathbb{N}$,*
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*Suppose $P(n)$ is a property of a natural number $n$.*
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We refer to the formula (†) as the *inductive hypothesis*.

**Proof.** Follows by mathematical induction, but using a property $Q(n)$ derived from $P(n)$. See the book. □

# *Example Proof Using Strong Induction*

**Proposition 1.2.5**

*Every nonempty set of natural numbers has a least element.*

**Proof.** Let $X$ be a nonempty set of natural numbers.

We begin by using strong induction to show that, for all $n \in \mathbb{N}$,

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if $m \in X$, then $X$ has a least element.

But $m \in X$, and thus $X$ has a least element. This completes our strong induction.

# *Example Proof (Cont.)*

**Proof (cont.).** Now we use the result of our strong induction to prove that $X$ has a least element. Since $X$ is a nonempty subset of $\mathbb{N}$, there is an $n \in \mathbb{N}$ such that $n \in X$. By the result of our induction, we can conclude that

$$\text{if } n \in X, \text{ then } X \text{ has a least element.}$$

But $n \in X$, and thus $X$ has a least element. □

# *Well-founded Induction*

We can also do induction over a well-founded relation.

A relation $R$ on a set $A$ is *well-founded* iff every nonempty subset $X$ of $A$ has an $R$-minimal element, where an element $x \in X$ is *$R$-minimal in $X$* iff there is no $y \in X$ such that $y \, R \, x$.

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Given $x, y \in A$, we say that *$y$ is a predecessor of $x$ in $R$* iff $y \, R \, x$. Thus $x \in X$ is $R$-minimal in $X$ iff none of $x$'s predecessors in $R$ (there may be none) are in $X$.

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For example, in Proposition 1.2.5, we proved that the strict total ordering $<$ on $\mathbb{N}$ is well-founded.

On the other hand, the strict total ordering $<$ on $\mathbb{Z}$ is *not* well-founded, as $\mathbb{Z}$ itself lacks a $<$-minimal element.

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Here's another negative example, showing that even if the underlying set is finite, the relation need not be well-founded.

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Let $A = \{0, 1\}$, and $R = \{(0, 1), (1, 0)\}$. Then is the only predecessor of $1$ in $R$, and is the only predecessor of $0$ in $R$.

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Here's another negative example, showing that even if the underlying set is finite, the relation need not be well-founded.

Let $A = \{0, 1\}$, and $R = \{(0, 1), (1, 0)\}$. Then $0$ is the only predecessor of $1$ in $R$, and is the only predecessor of $0$ in $R$.

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Of the nonempty subsets of $A$, we have that $\{0\}$ and $\{1\}$ have $R$-minimal elements. But consider $A$ itself. Then $0$ is not $R$-minimal in $A$, because $1 \in A$ and $1 \; R \; 0$. And $1$ is not $R$-minimal in $A$, because $0 \in A$ and $0 \; R \; 1$. Hence $R$ is not well-founded.

**Theorem 1.2.8 (Principle of Well-founded Induction)**
*Suppose $A$ is a set, $R$ is a well-founded relation on $A$, and $P(x)$ is a property of an element $x \in A$.*
*If*

> *for all $x \in A$,*
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We refer to the formula (†) as the *inductive hypothesis*.

When $A = \mathbb{N}$ and $R = <$, this is the same as the principle of strong induction.

# *Proof of Well-founded Induction*

**Proof.** Suppose $A$ is a set, $R$ is a well-founded relation on $A$, $P(x)$ is a property of an element $x \in A$, and

(‡) for all $x \in A$,
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Suppose, toward a contradiction, that it is not the case that, for all $x \in A$, $P(x)$. Hence there is an $x \in A$ such that $P(x)$ is false. Let $X = \{ x \in A \mid P(x) \text{ is false} \}$. Thus $x \in X$, showing that $X$ is non-empty. Because $R$ is well-founded on $A$, it follows that there is a $z \in X$ that is $R$-minimal in $X$, i.e., such that there is no $y \in X$ such that $y \, R \, z$.

# Proof of Well-founded Induction (Cont.)

**Proof (cont.).** By (‡), we have that

if for all $y \in A$, if $y\,R\,z$, then $P(y)$,
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# *Well-founded Induction on Predecessor Relation*

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$0$ has no predecessors in $\mathbf{pred}_{\mathbb{N}}$, and, for all $n \in \mathbb{N}$, $n$ is the only predecessor of $n+1$ in $\mathbf{pred}_{\mathbb{N}}$. Consequently, if a zero/non-zero case analysis is used, a proof by well-founded induction on $\mathbf{pred}_{\mathbb{N}}$ will look like a proof by

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# *Well-founded Induction on Integers via Absolute Value*

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If we do a well-founded induction on $R$, when proving $P(n)$, for $n \in \mathbb{Z}$, we can make use of $P(m)$ for any $m \in \mathbb{Z}$ whose absolute value is strictly less than the absolute value of $n$.

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E.g., when proving $P(-10)$, we could make use of $P(5)$ or $P(-9)$.