# *1.3: Inductive Definitions and Recursion*

In this section, we will introduce and study ordered trees of arbitrary (finite) arity, whose nodes are labeled by elements of some set.

In later chapters, we will define regular expressions (in Chapter 3), parse trees (in Chapter 4) and programs (in Chapter 5) as restrictions of the trees we consider here.

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The definition of the set of all trees is our first example of an inductive definition—a definition in which we collect together all of the values that can be constructed using a set of rules.

In this section, we will also see how to define functions by recursion.

# Inductive Definition of Trees

Suppose $X$ is a set. The set $\textbf{Tree}\,X$ of $X$-*trees* is the least set such that, for all $x \in X$ and $trs \in \textbf{List}(\textbf{Tree}\,X)$, $(x, trs) \in \textbf{Tree}\,X$.

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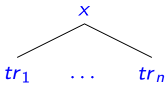
- For similar reasons, $(1, [\,])$ and $(6, [\,])$ are in $\textbf{Tree}\,\mathbb{N}$.
- Because $4 \in \mathbb{N}$, and $[(3, [\,]), (1, [\,]), (6, [\,])] \in \textbf{List}(\textbf{Tree}\,\mathbb{N})$, we have that $(4, [(3, [\,]), (1, [\,]), (6, [\,])]) \in \textbf{Tree}\,\mathbb{N}$.
- And we can continue like this forever.

# Drawing Trees

Trees are often easier to comprehend if they are drawn.

We draw the $X$-tree $(x, [tr_1, \ldots, tr_n])$ as

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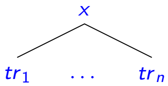


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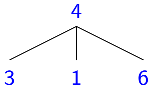
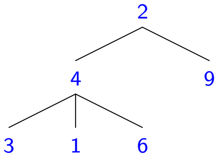



is the drawing of the $\mathbb{N}$-tree

$$(4, [(3, [\,]), (1, [\,]), (6, [\,])]).$$

# *Drawing Trees (Cont.)*

And

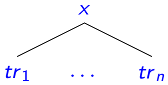



is the $\mathbb{N}$-tree

$$(2, [(4, [(3, [\,]), (1, [\,]), (6, [\,])]), (9, [\,])]).$$

# Tree Terminology and Notation

Consider the tree




again.

The *root label* of this tree is $x$, and $tr_1$ is the tree's *first child*, etc.
We write **rootLabel** $tr$ for the root label of $tr$.

# Tree Terminology and Notation (Cont.)

We often write a tree $(x, [tr_1, \ldots, tr_n])$ in a more compact, linear syntax:

- $x(tr_1, \ldots, tr_n)$, when $n \geq 1$, and
- $x$, when $n = 0$.

Thus

$$(2, [(4, [(3, [\,]), (1, [\,]), (6, [\,])]), (9, [\,])]).$$

can be written as

$$2(4(3, 1, 6), 9).$$

# Understanding the Definition

Consider the definition of $\mathbf{Tree}\, X$ again: the set $\mathbf{Tree}\, X$ of $X$-*trees* is the least set such that, (†) for all $x \in X$ and $trs \in \mathbf{List}(\mathbf{Tree}\, X)$, $(x, trs) \in \mathbf{Tree}\, X$.

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Let's call a set $U$ $X$-*closed* iff it satisfies property (†), where we have replaced $\textbf{Tree}\,X$ by $U$: for all $x \in X$ and $trs \in \textbf{List}\,U$, $(x, trs) \in U$.

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If $U$ is $\mathbb{Z}$-closed, then it will also be $\mathbb{N}$-closed. But if $\textbf{Tree}\,\mathbb{N}$ was $U$, it would have elements like $(-5, [\,])$, which are not wanted.

# *Understanding the Definition (Cont.)*

To keep **Tree** $X$ from having junk, we say that **Tree** $X$ is the set $U$ such that:

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But how do we know that such a set $U$ exists? (If $U$ and $U'$ are both $X$-closed sets that are subsets of all $X$-closed sets, then $U \subseteq U' \subseteq U$, and so $U = U'$. Thus there is at most one $U$ with the above property.)

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# *Justifying the Definition (Cont.)*

Let $\mathcal{W}$ be the set of all subsets of $V$ that are $X$-closed. Thus $\mathcal{W}$ is a nonempty set of $X$-closed sets, since $V \in \mathcal{W}$.

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Let $U = \bigcap \mathcal{W}$. Then $U$ is $X$-closed, and is a subset of all other $X$-closed sets (if $T$ is an $X$-closed set, then $T \cap V = \bigcap \{T, V\} \in \mathcal{W}$), i.e., it is the least $X$-closed set.

# *Principle of Induction on Trees*

Because trees are defined via an inductive definition, we get an induction principle for trees almost for free:

**Theorem 1.3.3 (Principle of Induction on Trees)**
*Suppose* $X$ *is a set and* $P(tr)$ *is a property of an element* $tr \in \mathbf{Tree}\, X$.
*If*

> *for all* $x \in X$ *and* $trs \in \mathbf{List}(\mathbf{Tree}\, X)$,
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We refer to (†) as the inductive hypothesis.

# *Proof of Principle of Induction on Trees*

**Proof.** Suppose $X$ is a set, $P(tr)$ is a property of an element $tr \in \mathbf{Tree}\, X$, and

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# Proof of Principle of Induction on Trees (Cont.)

**Proof (cont.).** Because $U$ is $X$-closed, we have that

□

## *Proof of Principle of Induction on Trees (Cont.)*

**Proof (cont.).** Because $U$ is $X$-closed, we have that $\mathbf{Tree}\,X \subseteq U$, as $\mathbf{Tree}\,X$ is the least $X$-closed set.

□

# *Proof of Principle of Induction on Trees (Cont.)*

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# *Destructing Trees*

**Proposition 1.3.4**

*Suppose $X$ is a set. For all $tr \in \textbf{Tree}\, X$, there are $x \in X$ and $trs \in \textbf{List}(\textbf{Tree}\, X)$ such that $tr = (x, trs)$.*

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## Predecessor Relation on Trees

Suppose $X$ is a set. Let the predecessor relation $\mathbf{pred}_{\mathbf{Tree}\,X}$ on $\mathbf{Tree}\,X$ be the set of all pairs of $X$-trees $(tr, tr')$ such that there are $x \in X$ and $trs' \in \mathbf{List}(\mathbf{Tree}\,X)$ such that $tr' = (x, trs')$ and $trs'\,i = tr$ for some $i \in [1 : |trs'|]$.

Thus the predecessors of a tree $(x, [tr_1, \ldots, tr_n])$ are its children $tr_1$, . . . , $tr_n$.

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**Proposition 1.3.5**

*If $X$ is a set, then $\mathbf{pred}_{\mathbf{Tree}\,X}$ is a well-founded relation on $\mathbf{Tree}\,X$.*

**Proof.** Suppose $X$ is a set and $Y$ is a nonempty subset of $\mathbf{Tree}\,X$. Mimicking Proposition 1.2.5, we can use the principle of induction on trees to prove that, for all $tr \in \mathbf{Tree}\,X$, if $tr \in Y$, then $Y$ has a $\mathbf{pred}_{\mathbf{Tree}\,X}$-minimal element. Because $Y$ is nonempty, we can conclude that $Y$ has a $\mathbf{pred}_{\mathbf{Tree}\,X}$-minimal element. □

# *Recursion*

Suppose $R$ is a well-founded relation on a set $A$. We can define a function $f$ from $A$ to a set $B$ by *well-founded recursion on* $R$.

The details are in the book, but the idea is simple: when $f$ is called with an element $x \in A$, it may call itself recursively on as many of the predecessors of $x$ in $R$ as it wants.

Typically, such a definition will be concrete enough that we can regard it as defining an algorithm as well as a function.

## *Examples of Well-founded Recursion*

- If we define $f \in \mathbb{N} \to B$ by well-founded recursion on $<$, then, when $f$ is called with $n \in \mathbb{N}$, it may call itself recursively on any strictly smaller natural numbers. In the case $n = 0$,

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- If we define $f \in \mathbb{N} \to B$ by well-founded recursion on the predecessor relation $\mathbf{pred}_{\mathbb{N}}$, then when $f$ is called with $n \in \mathbb{N}$, it may call itself recursively on $n - 1$, in the case when $n \geq 1$, and may make no recursive calls, when $n = 0$.

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  - $f\,0 = \cdots;$
  - for all $n \in \mathbb{N}$, $f(n + 1) = \cdots f\,n \cdots.$

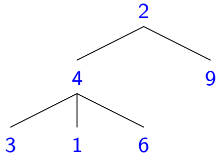
  We say that such a definition is by *recursion on* $\mathbb{N}$.

# *Examples of Well-founded Recursion (Cont.)*

- If we define $f \in \textbf{Tree}\,X \to B$ by well-founded recursion on the predecessor relation $\textbf{pred}_{\textbf{Tree}\,X}$, then when $f$ is called on an $X$-tree $(x, [tr_1, \ldots, tr_n])$, it may call itself recursively on any of $tr_1$, ..., $tr_n$. When $n = 0$, it may make no recursive calls. We say that such a definition is by *structural recursion*.

# *Examples of Well-founded Recursion (Cont.)*

- For example, we may define the *size* of an $X$-tree $(x, [tr_1, \ldots, tr_n])$ by summing the recursively computed sizes of $tr_1$, . . . , $tr_n$, and then adding $1$. Then, e.g., the size of

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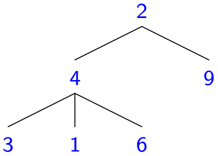


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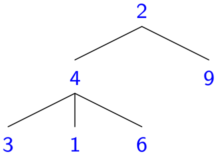


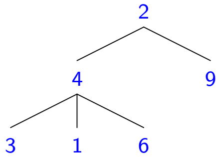

  is $6$.

  This defines a function $\textbf{size} \in \textbf{Tree}\, X \to \mathbb{N}$.

## *Examples of Well-founded Recursion (Cont.)*

- And we may define the *height* of an $X$-tree $(x, [tr_1, \ldots, tr_n])$ as
  - $0$, when $n = 0$, and
  - $1$ plus the maximum of the recursively computed heights of $tr_1$, . . . , $tr_n$, when $n \geq 1$.

  E.g., the height of

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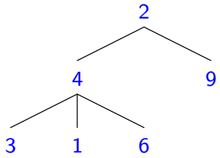


  is $2$.

# *Examples of Well-founded Recursion (Cont.)*

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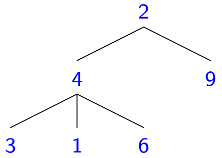


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  This defines a function **height** $\in$ **Tree** $X \to \mathbb{N}$.

## *Examples of Well-founded Recursion (Cont.)*

- Given a set $X$, we can define a well-founded relation $\textbf{size}_{\textbf{Tree}\,X}$ on $\textbf{Tree}\,X$ by: for all $tr, tr' \in \textbf{Tree}\,X$, $tr\ \textbf{size}_{\textbf{Tree}\,X}\ tr'$ iff $\textbf{size}\ tr < \textbf{size}\ tr'$.

  If we define a function $f \in \textbf{Tree}\,X \to B$ by well-founded recursion on $\textbf{size}_{\textbf{Tree}\,X}$, when $f$ is called with an $X$-tree $tr$, it may call itself recursively on any $X$-trees with strictly smaller sizes.

## *Examples of Well-founded Recursion (Cont.)*

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  If we define a function $f \in \textbf{Tree}\,X \to B$ by well-founded recursion on $\textbf{height}_{\textbf{Tree}\,X}$, when $f$ is called with an $X$-tree $tr$, it may call itself recursively on any $X$-trees with strictly smaller heights.

# *Examples of Well-founded Recursion (Cont.)*

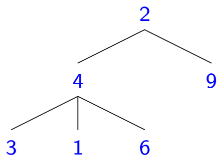
- Given a set $X$, we can define a well-founded relation $\mathbf{length}_{\mathbf{List}\,X}$ on $\mathbf{List}\,X$ by: for all $xs, ys \in \mathbf{List}\,X$, $xs\ \mathbf{length}_{\mathbf{List}\,X}\ ys$ iff $|xs| < |ys|$.

  If we define a function $f \in \mathbf{List}\,X \to B$ by well-founded recursion on $\mathbf{length}_{\mathbf{List}\,X}$, when $f$ is called with an $X$-list $xs$, it may call itself recursively on any $X$-lists with strictly smaller lengths.

# *Paths in Trees*

We can think of an $\mathbb{N} - \{0\}$-list $[n_1, n_2, \ldots, n_m]$ as a *path* through an $X$-tree $tr$: one starts with $tr$ itself, goes to the $n_1$-th child of $tr$, selects the $n_2$-th child of that tree, etc., stopping when the list is exhausted.

Consider the $\mathbb{N}$-tree




Then:

- $[\,]$ takes us to
- $[1]$ takes us to
- $[1, 3]$ takes us to
- $[1, 4]$ takes us to

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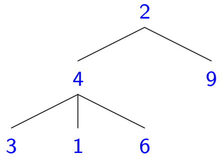


Then:

- $[\,]$ takes us to the whole tree.
- $[1]$ takes us to
- $[1, 3]$ takes us to
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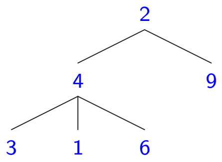


Then:

- $[\,]$ takes us to the whole tree.
- $[1]$ takes us to the tree $4(3, 1, 6)$.
- $[1, 3]$ takes us to
- $[1, 4]$ takes us to

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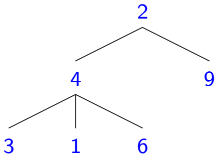


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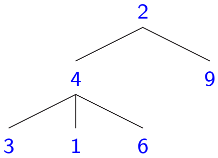


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- $[\,]$ takes us to the whole tree.
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- $[1, 3]$ takes us to the tree $6$.
- $[1, 4]$ takes us to no tree.

## *Paths in Trees*

We say that $xs \in \mathbf{List}(\mathbb{N} - \{0\})$ *is a valid path for* an $X$-tree $tr$ iff following the directions of $xs$ takes us from the top of $tr$ to some tree.

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For example, replacing the subtree at position $[1, 2]$ in $4(3(2, 1(7)), 6)$ by $3(7, 8)$ gives us $4(3(2, 3(7, 8)), 6)$.