3.10: Nondeterministic Finite Automata

In this section, we study the second of our more restricted kinds of finite automata: nondeterministic finite automata.

Definition of NFAs

A *nondeterministic finite automaton* (NFA) M is a finite automaton such that

$$T_M \subseteq \{\, q, x \to r \mid q, r \in \textbf{Sym} \text{ and } x \in \textbf{Str} \text{ and } |x| = 1 \,\}.$$

For example, $\mathsf{A}, 1 \to \mathsf{B}$ is a legal NFA transition, but $\mathsf{A}, \% \to \mathsf{B}$ and $\mathsf{A}, 11 \to \mathsf{B}$ are not legal.

We write **NFA** for the set of all nondeterministic finite automata. Thus **NFA** $\subsetneq$ **EFA** $\subsetneq$ **FA**.

Properties of NFAs

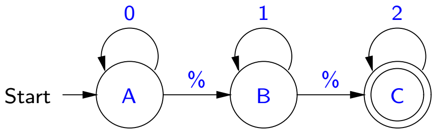
The following proposition obviously holds.

Proposition 3.10.1
Suppose M is an NFA.

- *For all $N \in$ **FA**, if M **iso** N, then N is an NFA.*
- *For all bijections f from Q_M to some set of symbols, **renameStates**(M, f) is an NFA.*
- ***renameStatesCanonically** M is an NFA.*
- ***simplify** M is an NFA.*

Converting EFAs to NFAs

Suppose M is the EFA



To convert M into an equivalent NFA, we will have to:

Converting EFAs to NFAs

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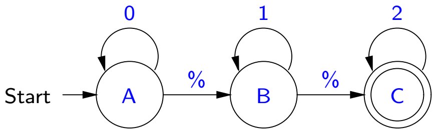


To convert M into an equivalent NFA, we will have to:

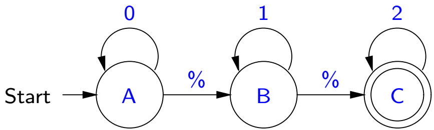
- replace the transitions $\mathsf{A}, \% \to \mathsf{B}$ and $\mathsf{B}, \% \to \mathsf{C}$ with legal transitions (for example, because of the valid labeled path

$$\mathsf{A} \overset{\%}{\Rightarrow} \mathsf{B} \overset{1}{\Rightarrow} \mathsf{B} \overset{\%}{\Rightarrow} \mathsf{C},$$

  we will add the transition $\mathsf{A}, 1 \to \mathsf{C}$);

Converting EFAs to NFAs

Suppose M is the EFA



To convert M into an equivalent NFA, we will have to:

- replace the transitions $\mathsf{A}, \% \rightarrow \mathsf{B}$ and $\mathsf{B}, \% \rightarrow \mathsf{C}$ with legal transitions (for example, because of the valid labeled path

$$\mathsf{A} \overset{\%}{\Rightarrow} \mathsf{B} \overset{1}{\Rightarrow} \mathsf{B} \overset{\%}{\Rightarrow} \mathsf{C},$$

  we will add the transition $\mathsf{A}, 1 \rightarrow \mathsf{C}$);
- make (at least) A be an accepting state (so that $\%$ is accepted by the NFA).

The Empty-closure of a Set of States

Suppose M is a finite automaton and $P \subseteq Q_M$. The *empty-closure* of P (**emptyClose**$_M$ P) is the least subset X of Q_M such that

- $P \subseteq X$;
- for all $q, r \in Q_M$, if $q \in X$ and $q, \% \to r \in T_M$, then $r \in X$.

The Empty-closure of a Set of States

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For example, if M is our example EFA and $P = \{\mathsf{A}\}$, then:

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For example, if M is our example EFA and $P = \{\mathsf{A}\}$, then:

- $\mathsf{A} \in X$;

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For example, if M is our example EFA and $P = \{\mathsf{A}\}$, then:

- $\mathsf{A} \in X$;
- $\mathsf{B} \in X$, since $\mathsf{A} \in X$ and $\mathsf{A}, \% \to \mathsf{B} \in T_M$;

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For example, if M is our example EFA and $P = \{\mathsf{A}\}$, then:

- $\mathsf{A} \in X$;
- $\mathsf{B} \in X$, since $\mathsf{A} \in X$ and $\mathsf{A}, \% \rightarrow \mathsf{B} \in T_M$;
- $\mathsf{C} \in X$, since $\mathsf{B} \in X$ and $\mathsf{B}, \% \rightarrow \mathsf{C} \in T_M$.

Thus $\textbf{emptyClose}\, P = \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$.

Backwards Empty-closure

Suppose M is a finite automaton and $P \subseteq Q_M$. The *backwards empty-closure* of P ($\textbf{emptyCloseBackwards}_M\, P$) is the least subset X of Q_M such that

- $P \subseteq X$;
- for all $q, r \in Q_M$, if $r \in X$ and $q, \% \to r \in T_M$, then $q \in X$.

Backwards Empty-closure

Suppose M is a finite automaton and $P \subseteq Q_M$. The *backwards empty-closure* of P (**emptyCloseBackwards**$_M$ P) is the least subset X of Q_M such that

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- $\mathsf{C} \in X$;

Backwards Empty-closure

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For example, if M is our example EFA and $P = \{\mathsf{C}\}$, then:

- $\mathsf{C} \in X$;
- $\mathsf{B} \in X$, since $\mathsf{C} \in X$ and $\mathsf{B}, \% \to \mathsf{C} \in T_M$;

Backwards Empty-closure

Suppose M is a finite automaton and $P \subseteq Q_M$. The *backwards empty-closure* of P (**emptyCloseBackwards**$_M$ P) is the least subset X of Q_M such that

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- for all $q, r \in Q_M$, if $r \in X$ and $q, \% \to r \in T_M$, then $q \in X$.

For example, if M is our example EFA and $P = \{\mathsf{C}\}$, then:

- $\mathsf{C} \in X$;
- $\mathsf{B} \in X$, since $\mathsf{C} \in X$ and $\mathsf{B}, \% \to \mathsf{C} \in T_M$;
- $\mathsf{A} \in X$, since $\mathsf{B} \in X$ and $\mathsf{A}, \% \to \mathsf{B} \in T_M$.

Thus **emptyCloseBackwards** $P = \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$.

Properies of Empty-closure and Backwards Empty-closure

Proposition 3.10.2
Suppose M is a finite automaton. For all $P \subseteq Q_M$,
$\textbf{emptyClose}_M\, P = \Delta_M(P, \quad)$.

Properies of Empty-closure and Backwards Empty-closure

Proposition 3.10.2

Suppose M is a finite automaton. For all $P \subseteq Q_M$, $\mathbf{emptyClose}_M\, P = \Delta_M(P, \%)$.

Properies of Empty-closure and Backwards Empty-closure

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Suppose M is a finite automaton. For all $P \subseteq Q_M$, $\textbf{emptyCloseBackwards}_M\, P = \{\, q \in Q_M \mid \Delta_M(\{q\}, \%) \cap$.

Properies of Empty-closure and Backwards Empty-closure

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Proposition 3.10.3

Suppose M is a finite automaton. For all $P \subseteq Q_M$, $\mathbf{emptyCloseBackwards}_M\, P = \{\, q \in Q_M \mid \Delta_M(\{q\}, \%) \cap P \neq \emptyset \,\}$.

Conversion Algorithm

We define a function/algorithm **efaToNFA** $\in$ **EFA** $\to$ **NFA** that converts EFAs into NFAs by saying that **efaToNFA** M is the NFA N such that:

- $Q_N = Q_M$;
- $s_N = s_M$;
- $A_N =$
- T_N is the set of all transitions $q', a \to r'$ such that $q', r' \in Q_M$, $a \in$ **Sym**, and there are $q, r \in Q_M$ such that:
  - $q, a \to r \in T_M$;
  - $q' \in$
  - $r' \in$

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  - $q, a \to r \in T_M$;
  - $q' \in$ **emptyCloseBackwards** $\{q\}$; and
  - $r' \in$ **emptyClose** $\{r\}$.

Conversion Algorithm

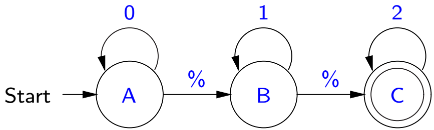
To compute the set T_N, we process each transition $q, x \to r$ of M as follows. If $x = \%$, then we generate no transitions. Otherwise, our transition is $q, a \to r$ for some symbol a. We then compute the backwards empty-closure of $\{q\}$, and call the result X, and compute the (forwards) empty-closure of $\{r\}$, and call the result Y. We then add all of the elements of

$$\{\, q', a \to r' \mid q' \in X \text{ and } r' \in Y \,\}$$

to T_N.

Conversion Example

Let M be our example EFA



and let $N =$ **efaToNFA** M. Then

- $Q_N = Q_M = \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$;
- $s_N = s_M = \mathsf{A}$;
- $A_N =$ **emptyCloseBackwards** $A_M =$ **emptyCloseBackwards** $\{\mathsf{C}\} = \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$.

Conversion Example

Now, let's work out what T_N is, by processing each of M's transitions.

- From the transitions $\mathsf{A}, \% \to \mathsf{B}$ and $\mathsf{B}, \% \to \mathsf{C}$, we get no elements of T_N.

Conversion Example

Now, let's work out what T_N is, by processing each of M's transitions.

- From the transitions $\mathsf{A}, \% \to \mathsf{B}$ and $\mathsf{B}, \% \to \mathsf{C}$, we get no elements of T_N.
- Consider the transition $\mathsf{A}, 0 \to \mathsf{A}$. Since **emptyCloseBackwards** $\{\mathsf{A}\} = \{\mathsf{A}\}$ and **emptyClose** $\{\mathsf{A}\} = \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$, we add

  to T_N.

Conversion Example

Now, let's work out what T_N is, by processing each of M's transitions.

- From the transitions $\mathsf{A}, \% \to \mathsf{B}$ and $\mathsf{B}, \% \to \mathsf{C}$, we get no elements of T_N.
- Consider the transition $\mathsf{A}, 0 \to \mathsf{A}$. Since **emptyCloseBackwards** $\{\mathsf{A}\} = \{\mathsf{A}\}$ and **emptyClose** $\{\mathsf{A}\} = \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$, we add $\mathsf{A}, 0 \to \mathsf{A}$, $\mathsf{A}, 0 \to \mathsf{B}$ and $\mathsf{A}, 0 \to \mathsf{C}$ to T_N.

Conversion Example

Now, let's work out what T_N is, by processing each of M's transitions.

- From the transitions $\mathsf{A}, \% \to \mathsf{B}$ and $\mathsf{B}, \% \to \mathsf{C}$, we get no elements of T_N.
- Consider the transition $\mathsf{A}, 0 \to \mathsf{A}$. Since **emptyCloseBackwards** $\{\mathsf{A}\} = \{\mathsf{A}\}$ and **emptyClose** $\{\mathsf{A}\} = \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$, we add $\mathsf{A}, 0 \to \mathsf{A}$, $\mathsf{A}, 0 \to \mathsf{B}$ and $\mathsf{A}, 0 \to \mathsf{C}$ to T_N.
- Consider the transition $\mathsf{B}, 1 \to \mathsf{B}$. Since **emptyCloseBackwards** $\{\mathsf{B}\} = \{\mathsf{A}, \mathsf{B}\}$ and **emptyClose** $\{\mathsf{B}\} = \{\mathsf{B}, \mathsf{C}\}$, we add

  to T_N.

Conversion Example

Now, let's work out what T_N is, by processing each of M's transitions.

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- Consider the transition $\mathsf{A}, 0 \to \mathsf{A}$. Since **emptyCloseBackwards** $\{\mathsf{A}\} = \{\mathsf{A}\}$ and **emptyClose** $\{\mathsf{A}\} = \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$, we add $\mathsf{A}, 0 \to \mathsf{A}$, $\mathsf{A}, 0 \to \mathsf{B}$ and $\mathsf{A}, 0 \to \mathsf{C}$ to T_N.
- Consider the transition $\mathsf{B}, 1 \to \mathsf{B}$. Since **emptyCloseBackwards** $\{\mathsf{B}\} = \{\mathsf{A}, \mathsf{B}\}$ and **emptyClose** $\{\mathsf{B}\} = \{\mathsf{B}, \mathsf{C}\}$, we add $\mathsf{A}, 1 \to \mathsf{B}$, $\mathsf{A}, 1 \to \mathsf{C}$, $\mathsf{B}, 1 \to \mathsf{B}$ and $\mathsf{B}, 1 \to \mathsf{C}$ to T_N.

Conversion Example

- Consider the transition $\mathsf{C}, 2 \to \mathsf{C}$. Since **emptyCloseBackwards** $\{\mathsf{C}\} = \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$ and **emptyClose** $\{\mathsf{C}\} = \{\mathsf{C}\}$, we add

  to T_N.

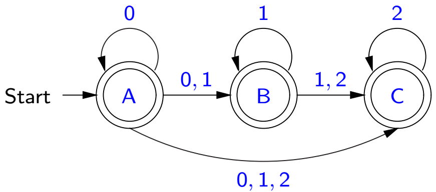
Conversion Example

- Consider the transition $\mathsf{C}, 2 \rightarrow \mathsf{C}$. Since **emptyCloseBackwards** $\{\mathsf{C}\} = \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$ and **emptyClose** $\{\mathsf{C}\} = \{\mathsf{C}\}$, we add $\mathsf{A}, 2 \rightarrow \mathsf{C}$, $\mathsf{B}, 2 \rightarrow \mathsf{C}$ and $\mathsf{C}, 2 \rightarrow \mathsf{C}$ to T_N.

Conversion Example

- Consider the transition $\mathsf{C}, 2 \to \mathsf{C}$. Since **emptyCloseBackwards** $\{\mathsf{C}\} = \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$ and **emptyClose** $\{\mathsf{C}\} = \{\mathsf{C}\}$, we add $\mathsf{A}, 2 \to \mathsf{C}$, $\mathsf{B}, 2 \to \mathsf{C}$ and $\mathsf{C}, 2 \to \mathsf{C}$ to T_N.

Thus our NFA N is



Specification of Conversion Function

Theorem 3.10.7
For all $M \in \mathbf{EFA}$*:*

- $\mathbf{efaToNFA}\, M \approx M$*; and*
- $\mathbf{alphabet}(\mathbf{efaToNFA}\, M) = \mathbf{alphabet}\, M$.

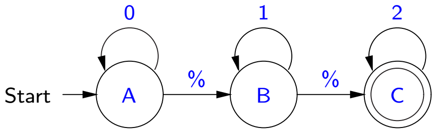
Empty-closure in Forlan

The Forlan module `FA` defines the following functions for computing forwards and backwards empty-closures:

```
val emptyClose          : fa -> sym set -> sym set
val emptyCloseBackwards : fa -> sym set -> sym set
```

Empty-closure in Forlan

For example, if `fa` is bound to the finite automaton



then we can compute the empty-closure of $\{A\}$ as follows:

```
- SymSet.output
= ("",
=  FA.emptyClose fa (SymSet.input ""));
@ A
@ .
A, B, C
val it = () : unit
```

Processing NFAs in Forlan

The Forlan module `NFA` defines an abstract type `nfa` (in the top-level environment) of nondeterministic finite automata, along with various functions for processing NFAs.

Values of type `nfa` are implemented as values of type `fa`, and the module NFA provides the following injection and projection functions:

```
val injToFA      : nfa -> fa
val injToEFA     : nfa -> efa
val projFromFA   : fa -> nfa
val projFromEFA  : efa -> nfa
```

The functions `injToFA`, `injToEFA`, `projFromFA` and `projFromEFA` are available in the top-level environment as `injNFAToFA`, `injNFAToEFA`, `projFATONFA` and `projEFATONFA`, respectively.

Processing NFAs in Forlan

The module `NFA` also defines the functions:

```
val input   : string -> nfa
val fromEFA : efa -> nfa
```

The function `input` is used to input an NFA, and the function `fromEFA` corresponds to our conversion function **efaToNFA**, and is available in the top-level environment with that name:

```
val efaToNFA : efa -> nfa
```

Processing NFAs in Forlan

Most of the functions for processing FAs that were introduced in previous sections are inherited by `NFA`:

```
val output                    : string * nfa -> unit
val numStates                 : nfa -> int
val numTransitions            : nfa -> int
val alphabet                  : nfa -> sym set
val equal                     : nfa * nfa -> bool
val checkLP                   : nfa -> lp -> unit
val validLP                   : nfa -> lp -> bool
val isomorphism               : nfa * nfa * sym_rel -> bool
val findIsomorphism           : nfa * nfa -> sym_rel
val isomorphic                : nfa * nfa -> bool
val renameStates              : nfa * sym_rel -> nfa
val renameStatesCanonically : nfa -> nfa
```

Processing NFAs in Forlan

More inherited functions:

```
val processStr        : nfa -> sym set * str -> sym set
val accepted          : nfa -> str -> bool
val findLP            : nfa -> sym set * str * sym set -> lp
val findAcceptingLP : nfa -> str -> lp
val simplified        : nfa -> bool
val simplify          : nfa -> nfa
```

Processing NFAs in Forlan

More inherited functions:

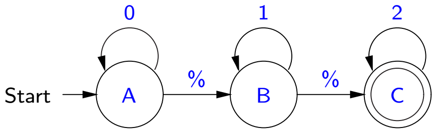
```
val processStr       : nfa -> sym set * str -> sym set
val accepted         : nfa -> str -> bool
val findLP           : nfa -> sym set * str * sym set -> lp
val findAcceptingLP : nfa -> str -> lp
val simplified       : nfa -> bool
val simplify         : nfa -> nfa
```

Finally, the functions for computing forwards and backwards empty-closures are inherited by the EFA module

```
val emptyClose           : efa -> sym set -> sym set
val emptyCloseBackwards : efa -> sym set -> sym set
```

Forlan Examples

Suppose that `efa` is the `efa`



Here are some example uses of a few of the above functions:

```
- projEFAToNFA efa;
invalid label in transition: "%"

uncaught exception Error
- val nfa = efaToNFA efa;
val nfa = - : nfa
```

Forlan Examples

```
- NFA.output("", nfa);
{states} A, B, C {start state} A
{accepting states} A, B, C
{transitions}
A, 0 -> A | B | C; A, 1 -> B | C; A, 2 -> C;
B, 1 -> B | C; B, 2 -> C; C, 2 -> C
val it = () : unit
```

Forlan Examples

```
- LP.output
= ("", EFA.findAcceptingLP efa (Str.input ""));
@ 012
@ .
A, 0 => A, % => B, 1 => B, % => C, 2 => C
val it = () : unit
- LP.output
= ("", NFA.findAcceptingLP nfa (Str.input ""));
@ 012
@ .
A, 0 => A, 1 => B, 2 => C
val it = () : unit
```