3.3: Simplification of Regular Expressions

In this section, we give three algorithms—of increasing power, but decreasing efficiency—for regular expression simplification.

The first algorithm—weak simplification—is defined via a straightforward structural recursion, and is sufficient for many purposes.

The remaining two algorithms—local simplification and global simplification—are based on a set of simplification rules that is still incomplete and evolving.

Regular Expression Complexity

To begin with, let's consider how we might measure the complexity/simplicity of regular expressions. The most obvious criterion is size (remember that regular expressions are trees). But consider this pair of equivalent regular expressions:

$$\alpha = (00^*11^*)^*, \text{ and}$$
$$\beta = \% + 0(0 + 11^*0)^*11^*.$$

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The standard measure of the closure-related complexity of a regular expression is its *star-height*: the maximum number $n \in \mathbb{N}$ such that there is a path from the root of the regular expression to one of its leaves that passes through n closures.

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α and β both have star-heights of 2.

Star-height isn't respected by the ways of forming regular expressions: 0 has strictly lower star-height than 0^*, but 01^* has the same star-height as 0^*1^*.

Closure Complexity

Let's define a *closure complexity* to be a nonempty list *ns* of natural numbers that is (not-necessarily strictly) descending.

E.g., $[3, 2, 2, 1]$ is a closure complexity, but $[3, 2, 3]$ and $[\,]$ are not.

This is a way of representing nonempty multisets of natural numbers.

We write **CC** for the set of all closure complexities.

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This is a way of representing nonempty multisets of natural numbers.

We write **CC** for the set of all closure complexities.

For all $n \in \mathbb{N}$, $[n]$ is a *singleton* closure complexity.

The *union* of closure complexities ns and ms ($ns \cup ms$) is the closure complexity that results from putting ns @ ms in descending order, keeping any duplicate elements. E.g., $[3, 2, 2, 1] \cup [4, 2, 1, 0] = [4, 3, 2, 2, 2, 1, 1, 0]$.

The *successor* $\overline{ns}$ of a closure complexity ns is the closure complexity formed by adding one to each element of ns, maintaining the order of the elements. E.g., $\overline{[3, 2, 2, 1]} = [4, 3, 3, 2]$.

Closure Complexity

Proposition 3.3.1

(1) For all $ns, ms \in \mathbf{CC}$, $ns \cup ms = ms \cup ns$.

(2) For all $ns, ms, ls \in \mathbf{CC}$, $(ns \cup ms) \cup ls = ns \cup (ms \cup ls)$.

(3) For all $ns, ms \in \mathbf{CC}$, $\overline{ns \cup ms} = \overline{ns} \cup \overline{ms}$.

Proposition 3.3.2

(1) For all $ns, ms \in \mathbf{CC}$, $\overline{ns} = \overline{ms}$ iff $ns = ms$.

(2) For all $ns, ms, ls \in \mathbf{CC}$, $ns \cup ls = ms \cup ls$ iff $ns = ms$.

Closure Complexity

We define a relation $<_{cc}$ on **CC** by: for all $ns, ms \in \mathbf{CC}$, $ns <_{cc} ms$ iff either:

- $ms = ns \mathbin{@} ls$ for some $ls \in \mathbf{CC}$; or
- there is an $i \in \mathbb{N} - \{0\}$ such that
  - $i \leq |ns|$ and $i \leq |ms|$,
  - for all $j \in [1 : i - 1]$, $ns\,j = ms\,j$, and
  - $ns\,i < ms\,i$.

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  - $i \leq |ns|$ and $i \leq |ms|$,
  - for all $j \in [1 : i - 1]$, $ns\,j = ms\,j$, and
  - $ns\,i < ms\,i$.

E.g., $[2, 2] <_{cc} [2, 2, 1]$ and $[2, 1, 1, 0, 0] <_{cc} [2, 2, 1]$.

Proposition 3.3.3

(1) *For all* $ns, ms \in \mathbf{CC}$, $\overline{ns} <_{cc} \overline{ms}$ *iff* $ns <_{cc} ms$.

(2) *For all* $ns, ms, ls \in \mathbf{CC}$, $ns \cup ls <_{cc} ms \cup ls$ *iff* $ns <_{cc} ms$.

(3) *For all* $ns, ms \in \mathbf{CC}$, $ns <_{cc} ns \cup ms$.

Proposition 3.3.4

$<_{cc}$ *is a strict total ordering on* $\mathbf{CC}$.

Proposition 3.3.5

$<_{cc}$ *is a well-founded relation on* $\mathbf{CC}$.

Closure Complexity

Now we can define the closure complexity of a regular expression. Define the function $\mathbf{cc} \in \mathbf{Reg} \to \mathbf{CC}$ by structural recursion:

$$
\begin{aligned}
\mathbf{cc}\,\% &= [0];\\
\mathbf{cc}\,\$ &= [0];\\
\mathbf{cc}\,a &= [0], \text{ for all } a \in \mathbf{Sym};\\
\mathbf{cc}(*(\alpha)) &= \overline{\mathbf{cc}\,\alpha}, \text{ for all } \alpha \in \mathbf{Reg};\\
\mathbf{cc}(@(\alpha,\beta)) &= \mathbf{cc}\,\alpha \cup \mathbf{cc}\,\beta, \text{ for all } \alpha,\beta \in \mathbf{Reg}; \text{ and}\\
\mathbf{cc}(+(\alpha,\beta)) &= \mathbf{cc}\,\alpha \cup \mathbf{cc}\,\beta, \text{ for all } \alpha,\beta \in \mathbf{Reg}.
\end{aligned}
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We say that $\mathbf{cc}\,\alpha$ is *the closure complexity of* α.

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\end{aligned}$$

We say that $\mathbf{cc}\,\alpha$ is *the closure complexity of* α.

E.g.,

$$\begin{aligned}
\mathbf{cc}((12^*)^*) &= \overline{\mathbf{cc}(12^*)} = \overline{\mathbf{cc}\,1 \cup \mathbf{cc}(2^*)} = \overline{[0] \cup \overline{\mathbf{cc}\,2}} \\
&= \overline{[0] \cup \overline{[0]}} = \overline{[0] \cup [1]} = \overline{[1,0]} = [2,1].
\end{aligned}$$

Closure Complexity

Returning to our initial examples, we have that
$\textbf{cc}((00^*11^*)^*) = [2, 2, 1, 1]$ and
$\textbf{cc}(\% + 0(0 + 11^*0)^*11^*) = [2, 1, 1, 1, 1, 0, 0, 0]$.

Since $[2, 1, 1, 1, 1, 0, 0, 0] <_{cc} [2, 2, 1, 1]$, the closure complexity of $\% + 0(0 + 11^*0)^*11^*$ is strictly smaller than the closure complexity of $(00^*11^*)^*$.

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Proposition 3.3.6
For all $\alpha \in \mathbf{Reg}$, $|\mathbf{cc}\,\alpha| =$.

Proof. An easy induction on regular expressions. □

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Since $[2,1,1,1,1,0,0,0] <_{cc} [2,2,1,1]$, the closure complexity of $\% + 0(0 + 11^*0)^*11^*$ is strictly smaller than the closure complexity of $(00^*11^*)^*$.

Proposition 3.3.6
For all $\alpha \in \mathbf{Reg}$, $|\mathbf{cc}\,\alpha| = \mathbf{numLeaves}\,\alpha$.

Proof. An easy induction on regular expressions. □

Proposition 3.3.9
Suppose $\alpha, \beta, \beta' \in \textbf{Reg}$, $\textbf{cc}\,\beta = \textbf{cc}\,\beta'$, *pat* $\in$ **Path** *is valid for* α, *and* β *is the subtree of* α *at position pat. Let* α' *be the result of replacing the subtree at position pat in* α *by* β'*. Then* $\textbf{cc}\,\alpha = \textbf{cc}\,\alpha'$.

Proof. By induction on α. □

Proposition 3.3.11
Suppose $\alpha, \beta, \beta' \in \textbf{Reg}$, $\textbf{cc}\,\beta' <_{cc} \textbf{cc}\,\beta$, *pat* $\in$ **Path** *is valid for* α, *and* β *is the subtree of* α *at position pat. Let* α' *be the result of replacing the subtree at position pat in* α *by* β'*. Then* $\textbf{cc}\,\alpha' <_{cc} \textbf{cc}\,\alpha$.

Proof. By induction on α. □

Regular Expression Complexity

When judging the relative complexities of regular expressions α and β, we will first look at how their closure complexities are related.

And, when their closure complexities are equal, we will look at how their sizes are related. To finish explaining how we will judge the relative complexity of regular expressions, we need three definitions.

Numbers of Concatenations and Symbols

We write **numConcats** α and **numSyms** α for the number of concatenations and symbols, respectively, in α.

E.g., **numConcats**$(((01)^*(01))^*) = 3$. and **numSyms**$((0^*1) + 0) = 3$.

Standardization

We say that a regular expression α is *standardized* iff none of α's subtrees have any of the following forms:

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- $(\beta_1\beta_2)\beta_3$ (concatenations should be grouped to the right); and
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If α is standardized, all of its subtrees are standardized.

Judging Relative Complexity

Returning to our assessment of regular expression complexity, suppose that α and β are regular expressions generating %. Then $(\alpha\beta)^*$ and $(\alpha + \beta)^*$ are equivalent, and have the same closure complexity and size,

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Consequently, when two regular expression have the same closure complexity and size, we will judge their relative complexity according to their numbers of concatenations.

Judging Relative Complexity

Next, consider the regular expressions $0 + 01$ and $0(\% + 1)$.

These regular expressions have the same closure complexity $[0, 0, 0]$, size (5) and number of concatenations (1).

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And we can base this preference on the fact that the number of symbols of $0(\% + 1)$ (2) is one less than the number of symbols of $0 + 01$.

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Next, consider the regular expressions $0 + 01$ and $0(\% + 1)$.

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And we can base this preference on the fact that the number of symbols of $0(\% + 1)$ (2) is one less than the number of symbols of $0 + 01$.

Thus, when regular expressions have identical closure complexity, size and number of concatenations, we will use their relative numbers of symbols to judge their relative complexity.

Judging Relative Complexity

Finally, when regular expressions have the same closure complexity, size, number of concatenations, and number of symbols, we will judge their relative complexity according to whether they are standardized, thinking that a standardized regular expression is simpler than one that is not standardized.

Judging Relative Complexity

We define a relation $<_{\textsf{simp}}$ on **Reg** by, for all $\alpha, \beta \in$ **Reg**, $\alpha <_{\textsf{simp}} \beta$ iff:

- **cc** $\alpha <_{cc}$ **cc** β; or

We read $\alpha <_{\textsf{simp}} \beta$ as α is *simpler* (less *complex*) than β.

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- **cc** $\alpha =$ **cc** β but **size** $\alpha <$ **size** β; or

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- **cc** $\alpha <_{cc}$ **cc** β; or
- **cc** $\alpha =$ **cc** β but **size** $\alpha <$ **size** β; or
- **cc** $\alpha =$ **cc** β and **size** $\alpha =$ **size** β, but **numConcats** $\alpha <$ **numConcats** β; or

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- **cc** $\alpha <_{cc}$ **cc** β; or
- **cc** $\alpha =$ **cc** β but **size** $\alpha <$ **size** β; or
- **cc** $\alpha =$ **cc** β and **size** $\alpha =$ **size** β, but **numConcats** $\alpha <$ **numConcats** β; or
- **cc** $\alpha =$ **cc** β, **size** $\alpha =$ **size** β and **numConcats** $\alpha =$ **numConcats** β, but **numSyms** $\alpha <$ **numSyms** β; or

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- **cc** $\alpha =$ **cc** β and **size** $\alpha =$ **size** β, but **numConcats** $\alpha <$ **numConcats** β; or
- **cc** $\alpha =$ **cc** β, **size** $\alpha =$ **size** β and **numConcats** $\alpha =$ **numConcats** β, but **numSyms** $\alpha <$ **numSyms** β; or
- **cc** $\alpha =$ **cc** β, **size** $\alpha =$ **size** β, **numConcats** $\alpha =$ **numConcats** β and **numSyms** $\alpha =$ **numSyms** β, but α is standardized and β is not standardized.

We read $\alpha <_{\mathsf{simp}} \beta$ as α is *simpler* (less *complex*) than β.

Judging Relative Complexity

We define a relation $\equiv_{\textsf{simp}}$ on **Reg** by, for all $\alpha, \beta \in$ **Reg**,
$\alpha \equiv_{\textsf{simp}} \beta$ iff α and β have the same closure complexity, size, numbers of concatenations, numbers of symbols, and status of being (or not being) standardized.

We read $\alpha \equiv_{\textsf{simp}} \beta$ as α and β have the *same complexity*.

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We read $\alpha \equiv_{\textsf{simp}} \beta$ as α and β have the *same complexity*.

We define a relation $\leq_{\textsf{simp}}$ on **Reg** by, for all $\alpha, \beta \in$ **Reg**, $\alpha \leq_{\textsf{simp}} \beta$ iff $\alpha <_{\textsf{simp}} \beta$ or $\alpha \equiv_{\textsf{simp}} \beta$.

We read $\alpha \leq_{\textsf{simp}} \beta$ as α is *at least as simple as* (*no more complex than*) β.

Judging Relative Complexity

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We define a relation $\leq_{\textsf{simp}}$ on **Reg** by, for all $\alpha, \beta \in$ **Reg**, $\alpha \leq_{\textsf{simp}} \beta$ iff $\alpha <_{\textsf{simp}} \beta$ or $\alpha \equiv_{\textsf{simp}} \beta$.

We read $\alpha \leq_{\textsf{simp}} \beta$ as α is *at least as simple as* (*no more complex than*) β.

For example, the following regular expressions are equivalent and have the same complexity:

$$1(01 + 10) + (\% + 01)1 \quad \text{and} \quad 011 + 1(\% + 01 + 10).$$

Proposition 3.3.12

(1) $<_{\textsf{simp}}$ *is transitive.*

(2) $\equiv_{\textsf{simp}}$ *is reflexive on* **Reg**, *transitive and symmetric.*

(3) For all $\alpha, \beta \in$ **Reg**, *exactly one of the following holds:* $\alpha <_{\textsf{simp}} \beta$, $\beta <_{\textsf{simp}} \alpha$ *or* $\alpha \equiv_{\textsf{simp}} \beta$.

(4) $\leq_{\textsf{simp}}$ *is transitive, and, for all* $\alpha, \beta \in$ **Reg**, $\alpha \equiv_{\textsf{simp}} \beta$ *iff* $\alpha \leq_{\textsf{simp}} \beta$ *and* $\beta \leq_{\textsf{simp}} \alpha$.

Closure Complexity in Forlan

The Forlan module `Reg` defines the abstract type `cc` of closure complexities, along with these functions:

```
val ccToList  : cc -> int list
val singCC    : int -> cc
val unionCC   : cc * cc -> cc
val succCC    : cc -> cc
val cc        : reg -> cc
val compareCC : cc * cc -> order
```

Closure Complexity in Forlan

Here are some examples of how these functions can be used:

```
- val ns =
=       Reg.succCC
=       (Reg.unionCC(Reg.singCC 1, Reg.singCC 1));
val ns = - : Reg.cc
- Reg.ccToList ns;
val it = [2,2] : int list
- val ms = Reg.unionCC(ns, Reg.succCC ns);
val ms = - : Reg.cc
- Reg.ccToList ms;
val it = [3,3,2,2] : int list
```

Closure Complexity in Forlan

```
- Reg.ccToList(Reg.cc(Reg.fromString "(00*11*)*"));
val it = [2,2,1,1] : int list
- Reg.ccToList
= (Reg.cc(Reg.fromString "% + 0(0 + 11*0)*11*"));
val it = [2,1,1,1,1,0,0,0] : int list
- Reg.compareCC
= (Reg.cc(Reg.fromString "(00*11*)*"),
=  Reg.cc(Reg.fromString "% + 0(0 + 11*0)*11*"));
val it = GREATER : order
- Reg.compareCC
= (Reg.cc(Reg.fromString "(00*11*)*"),
=  Reg.cc(Reg.fromString "(1*10*0)*"));
val it = EQUAL : order
```

Regular Expression Complexity in Forlan

The module Reg also includes these functions:

```
val numConcats        : reg -> int
val numSyms           : reg -> int
val standardized      : reg -> bool
val compareComplexity : reg * reg -> order
```

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val numConcats         : reg -> int
val numSyms            : reg -> int
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```

Here are some examples of how these functions can be used:

```
- Reg.numConcats(Reg.fromString "(01)*(10)*");
val it = 3 : int
- Reg.numSyms(Reg.fromString "(01)*(10)*");
val it = 4 : int
- Reg.standardized(Reg.fromString "00*1");
val it = true : bool
- Reg.standardized(Reg.fromString "00*0");
val it = false : bool
```

Regular Expression Complexity in Forlan

```
- Reg.compareComplexity
= (Reg.fromString "(00*11*)*",
=  Reg.fromString "% + 0(0 + 11*0)*11*");
val it = GREATER : order
- Reg.compareComplexity
= (Reg.fromString "0**1**", Reg.fromString "(01)**");
val it = GREATER : order
- Reg.compareComplexity
= (Reg.fromString "(0*1*)*",
=  Reg.fromString "(0*+1*)*");
val it = GREATER : order
- Reg.compareComplexity
= (Reg.fromString "0+01", Reg.fromString "0(%+1)");
val it = GREATER : order
- Reg.compareComplexity
= (Reg.fromString "(01)2", Reg.fromString "012");
val it = GREATER : order
```

Regular Expression Complexity in Forlan

```
- Reg.compareComplexity
= (Reg.fromString "1(01+10)+(%+01)1",
=  Reg.fromString "011+1(%+01+10)");
val it = EQUAL : order
```

Weak Simplification

We say that a regular expression α is *weakly simplified* iff α is standardized and none of α's subtrees have any of the following forms:

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- $\beta + \beta$ or $\beta + (\beta + \gamma)$ (the duplicate occurrence of β is redundant);

Weak Simplification

We say that a regular expression α is *weakly simplified* iff α is standardized and none of α's subtrees have any of the following forms:

- $\$ + \beta$ or $\beta + \$$ (the $\$$ is redundant);
- $\beta + \beta$ or $\beta + (\beta + \gamma)$ (the duplicate occurrence of β is redundant);
- $\%\beta$ or $\beta\%$ (the $\%$ is redundant);

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- $\%^*$ or $\$^*$ or $(\beta^*)^*$ (the first two can be replaced by $\%$, and the extra closure can be omitted in the third case).

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- $\$\beta$ or $\beta\$$ (both are equivalent to $\$$); and
- $\%^*$ or $\$^*$ or $(\beta^*)^*$ (the first two can be replaced by $\%$, and the extra closure can be omitted in the third case).

If α is weakly simplified, all of its subtrees are weakly simplified.

Proposition 3.3.13

(1) *For all* $\alpha \in \textbf{Reg}$, *if* α *is weakly simplified and* $L(\alpha) = \emptyset$, *then* $\alpha =$

Weak Simplification

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Proof.

Weak Simplification

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Proof. The three parts are proved in order, using induction on regular expressions.

Weak Simplification

Proposition 3.3.13

(1) *For all $\alpha \in$ **Reg**, if α is weakly simplified and $L(\alpha) = \emptyset$, then $\alpha = \$$.*

(2) *For all $\alpha \in$ **Reg**, if α is weakly simplified and $L(\alpha) = \{\%\}$, then $\alpha = \%$.*

(3) *For all $\alpha \in$ **Reg**, for all $a \in$ **Sym**, if α is weakly simplified and $L(\alpha) = \{a\}$, then $\alpha = a$.*

Proof. The three parts are proved in order, using induction on regular expressions.

For part (3), suppose $a \in$ **Sym**. It suffices to show that, for all $\alpha \in$ **Reg**,

if α is weakly simplified and $L(\alpha) = \{a\}$, then $\alpha = a$.

We show the concatenation case.

Proof (cont.). Suppose $\alpha, \beta \in$ **Reg** and assume the inductive hypothesis:

Weak Simplification

Proof (cont.). Suppose $\alpha, \beta \in \mathbf{Reg}$ and assume the inductive hypothesis: if α is weakly simplified and $L(\alpha) = \{a\}$, then $\alpha = a$; and if β is weakly simplified and $L(\beta) = \{a\}$, then $\beta = a$.

Weak Simplification

Proof (cont.). Suppose $\alpha, \beta \in$ **Reg** and assume the inductive hypothesis: if α is weakly simplified and $L(\alpha) = \{a\}$, then $\alpha = a$; and if β is weakly simplified and $L(\beta) = \{a\}$, then $\beta = a$. Assume that $\alpha\beta$ is weakly simplified and $L(\alpha\beta) = \{a\}$. We must show that $\alpha\beta = a$.

Weak Simplification

Proof (cont.). Suppose $\alpha, \beta \in$ **Reg** and assume the inductive hypothesis: if α is weakly simplified and $L(\alpha) = \{a\}$, then $\alpha = a$; and if β is weakly simplified and $L(\beta) = \{a\}$, then $\beta = a$. Assume that $\alpha\beta$ is weakly simplified and $L(\alpha\beta) = \{a\}$. We must show that $\alpha\beta = a$. We have that α and β are weakly simplified. Since $L(\alpha)L(\beta) = L(\alpha\beta) = \{a\}$, we have two cases to consider.

Weak Simplification

Proof (cont.). Suppose $\alpha, \beta \in$ **Reg** and assume the inductive hypothesis: if α is weakly simplified and $L(\alpha) = \{a\}$, then $\alpha = a$; and if β is weakly simplified and $L(\beta) = \{a\}$, then $\beta = a$. Assume that $\alpha\beta$ is weakly simplified and $L(\alpha\beta) = \{a\}$. We must show that $\alpha\beta = a$. We have that α and β are weakly simplified. Since $L(\alpha)L(\beta) = L(\alpha\beta) = \{a\}$, we have two cases to consider.

- Suppose $L(\alpha) = \{a\}$ and $L(\beta) = \{\%\}$.

- Suppose $L(\alpha) = \{\%\}$ and $L(\beta) = \{a\}$. The proof of this case is similar to that of the other one.

Weak Simplification

Proof (cont.). Suppose $\alpha, \beta \in$ **Reg** and assume the inductive hypothesis: if α is weakly simplified and $L(\alpha) = \{a\}$, then $\alpha = a$; and if β is weakly simplified and $L(\beta) = \{a\}$, then $\beta = a$. Assume that $\alpha\beta$ is weakly simplified and $L(\alpha\beta) = \{a\}$. We must show that $\alpha\beta = a$. We have that α and β are weakly simplified. Since $L(\alpha)L(\beta) = L(\alpha\beta) = \{a\}$, we have two cases to consider.

- Suppose $L(\alpha) = \{a\}$ and $L(\beta) = \{\%\}$. Since β is weakly simplified and $L(\beta) = \{\%\}$, part (2) tells us that $\beta = \%$.

- Suppose $L(\alpha) = \{\%\}$ and $L(\beta) = \{a\}$. The proof of this case is similar to that of the other one.

Proof (cont.). Suppose $\alpha, \beta \in \mathbf{Reg}$ and assume the inductive hypothesis: if α is weakly simplified and $L(\alpha) = \{a\}$, then $\alpha = a$; and if β is weakly simplified and $L(\beta) = \{a\}$, then $\beta = a$. Assume that $\alpha\beta$ is weakly simplified and $L(\alpha\beta) = \{a\}$. We must show that $\alpha\beta = a$. We have that α and β are weakly simplified. Since $L(\alpha)L(\beta) = L(\alpha\beta) = \{a\}$, we have two cases to consider.

- Suppose $L(\alpha) = \{a\}$ and $L(\beta) = \{\%\}$. Since β is weakly simplified and $L(\beta) = \{\%\}$, part (2) tells us that $\beta = \%$. But this means that $\alpha\beta = \alpha\%$ is not weakly simplified after all—contradiction. Thus we can conclude that $\alpha\beta = a$.
- Suppose $L(\alpha) = \{\%\}$ and $L(\beta) = \{a\}$. The proof of this case is similar to that of the other one.

Weak Simplification

Proof (cont.). Suppose $\alpha, \beta \in \textbf{Reg}$ and assume the inductive hypothesis: if α is weakly simplified and $L(\alpha) = \{a\}$, then $\alpha = a$; and if β is weakly simplified and $L(\beta) = \{a\}$, then $\beta = a$. Assume that $\alpha\beta$ is weakly simplified and $L(\alpha\beta) = \{a\}$. We must show that $\alpha\beta = a$. We have that α and β are weakly simplified. Since $L(\alpha)L(\beta) = L(\alpha\beta) = \{a\}$, we have two cases to consider.

- Suppose $L(\alpha) = \{a\}$ and $L(\beta) = \{\%\}$. Since β is weakly simplified and $L(\beta) = \{\%\}$, part (2) tells us that $\beta = \%$. But this means that $\alpha\beta = \alpha\%$ is not weakly simplified after all—contradiction. Thus we can conclude that $\alpha\beta = a$.
- Suppose $L(\alpha) = \{\%\}$ and $L(\beta) = \{a\}$. The proof of this case is similar to that of the other one.

(Note that we didn't use the inductive hypothesis on either α or β.)

Proof (cont.). We use both parts of the inductive hypothesis when proving the

□

Weak Simplification

Proof (cont.). We use both parts of the inductive hypothesis when proving the union case. If $L(\alpha) \cup L(\beta) = L(\alpha + \beta) = \{a\}$, then one possibility is that one of $L(\alpha)$ or $L(\beta)$ is $\emptyset$, in which case we use part (1) to get our contradiction. Otherwise,

□

Proof (cont.). We use both parts of the inductive hypothesis when proving the union case. If $L(\alpha) \cup L(\beta) = L(\alpha + \beta) = \{a\}$, then one possibility is that one of $L(\alpha)$ or $L(\beta)$ is $\emptyset$, in which case we use part (1) to get our contradiction. Otherwise, $L(\alpha) = \{a\} = L(\beta)$, and so the inductive hypothesis tells us $\alpha = a = \beta$, so that $\alpha + \beta = a + a$, giving us the contradiction. □

Proposition 3.3.14

For all $\alpha \in$ **Reg**, *if* α *is weakly simplified, then* **alphabet**$(L(\alpha))$ **alphabet** α.

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For all $\alpha \in \textbf{Reg}$, *if* α *is weakly simplified, then* $\textbf{alphabet}(L(\alpha)) = \textbf{alphabet}\,\alpha$.

Weak Simplification

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For all $\alpha \in \textbf{Reg}$*, if* α *is weakly simplified and* α *has one or more occurrences of* $\$$*, then* $\alpha =$

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Proposition 3.3.15

For all $\alpha \in \mathbf{Reg}$, *if* α *is weakly simplified and* α *has one or more occurrences of* $\$$, *then* $\alpha = \$$.

Proposition 3.3.16

For all $\alpha \in \mathbf{Reg}$, *if* α *is weakly simplified and* α *has one or more closures, then* $L(\alpha)$ *is infinite.*

Weak Simplification

Let

$$\mathbf{WS} = \{\, \alpha \in \mathbf{Reg} \mid \alpha \text{ is weakly simplified} \,\}.$$

Weak Simplification

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$$\textbf{WS} = \{\, \alpha \in \textbf{Reg} \mid \alpha \text{ is weakly simplified} \,\}.$$

Define a function **deepClosure** $\in$ **WS** $\to$ **WS** as follows. For all $\alpha \in$ **WS**:

$$\begin{aligned}
\textbf{deepClosure}\,\% &= \%, \\
\textbf{deepClosure}\,\$ &= \%, \\
\textbf{deepClosure}\,(*(\alpha)) &= \alpha^*, \text{ and} \\
\textbf{deepClosure}\,\alpha &= \alpha^*, \text{ if } \alpha \notin \{\%, \$\} \text{ and } \alpha \text{ is not a closure.}
\end{aligned}$$

Weak Simplification

Define a function **deepConcat** $\in$ **WS** $\times$ **WS** $\to$ **WS** as follows. For all $\alpha, \beta \in$ **WS**:

$$\begin{aligned}
\textbf{deepConcat}(\alpha, \$) &= \$, \\
\textbf{deepConcat}(\$, \alpha) &= \$, \text{ if } \alpha \neq \$, \\
\textbf{deepConcat}(\alpha, \%) &= \alpha, \text{ if } \alpha \neq \$, \\
\textbf{deepConcat}(\%, \alpha) &= \alpha, \text{ if } \alpha \notin \{\$, \%\}, \text{ and} \\
\textbf{deepConcat}(\alpha, \beta) &= \textbf{shiftClosuresRight}(\textbf{rightConcat}(\alpha, \beta)), \\
&\qquad \text{if } \alpha, \beta \notin \{\$, \%\},
\end{aligned}$$

If α_n is not a concatenation, then **rightConcat**$(\alpha_1 \cdots \alpha_n, \beta) = \alpha_1 \cdots \alpha_n \beta$.

E.g., **rightConcat**$(00, 00)$ is 0000 $(0(0(00)))$, not $(00)(00)$.

Weak Simplification

Define a function **deepConcat** $\in$ **WS** $\times$ **WS** $\to$ **WS** as follows. For all $\alpha, \beta \in$ **WS**:

$$
\begin{aligned}
\textbf{deepConcat}(\alpha, \$) &= \$, \\
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\textbf{deepConcat}(\alpha, \%) &= \alpha, \ \text{if } \alpha \neq \$, \\
\textbf{deepConcat}(\%, \alpha) &= \alpha, \ \text{if } \alpha \notin \{\$, \%\}, \text{ and} \\
\textbf{deepConcat}(\alpha, \beta) &= \textbf{shiftClosuresRight}(\textbf{rightConcat}(\alpha, \beta)), \\
&\qquad \text{if } \alpha, \beta \notin \{\$, \%\},
\end{aligned}
$$

shiftClosuresRight repeatedly applies the following rules down the rightmost branch: $\beta^*\beta \to$ **rightConcat**(β, β^*), $\beta^*\beta\gamma \to \beta\beta^*\gamma$, $(\beta_1\beta_2)^*\beta_1 \to \beta_1($**rightConcat**$(\beta_2, \beta_1))^*$ and $(\beta_1\beta_2)^*\beta_1\gamma \to \beta_1($**rightConcat**$(\beta_2, \beta_1))^*\gamma$.

Define a function **deepUnion** $\in$ **WS** $\times$ **WS** $\to$ **WS** as follows. For all $\alpha, \beta \in$ **WS**:

deepUnion$(\alpha, \$) = \alpha,$

deepUnion$(\$, \alpha) = \alpha,$ if $\alpha \neq \$,$ and

deepUnion$(\alpha, \beta) =$ **sortUnions**(**rightUnion**$(\alpha, \beta)),$ if $\alpha \neq \$$ and $\beta \neq \$.$

If α_n is not a union, then
rightUnion$(\alpha_1 + \cdots + \alpha_n, \beta) = \alpha_1 + \cdots + \alpha_n + \beta$.

sortUnions sorts the unions down the right branch using our total ordering on **Reg**, removing duplicates.

Weak Simplification

Define **weaklySimplify** $\in$ **Reg** $\to$ **WS** by structural recursion:

- **weaklySimplify** % = %;
- **weaklySimplify** \$ = \$;
- **weaklySimplify** $a = a$, for all $a \in$ **Sym**;
- **weaklySimplify**$(*(\alpha))$ =

  deepClosure(**weaklySimplify** α);

- **weaklySimplify**(@(α, β)) =

  deepConcat(**weaklySimplify** α, **weaklySimplify** β); and

- **weaklySimplify**$(+(\alpha, \beta))$ =

  deepUnion(**weaklySimplify** α, **weaklySimplify** β).

Proposition 3.3.28

For all $\alpha \in$ **Reg***:*

(1) **weaklySimplify** $\alpha \approx \alpha$;

(2) **alphabet**(**weaklySimplify** α) $\subseteq$ **alphabet** α;

(3) **cc**(**weaklySimplify** α) $\leq_{cc}$ **cc** β;

(4) **size**(**weaklySimplify** α) $\leq$ **size** α;

(5) **numSyms**(**weaklySimplify** α) $\leq$ **numSyms** α; *and*

(6) **numConcats**(**weaklySimplify** α) $\leq$ **numConcats** α.

Proof. By induction on regular expressions. □

Proposition 3.3.28

For all $\alpha \in \textbf{Reg}$*:*

(1) $\textbf{weaklySimplify}\,\alpha \approx \alpha$;

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(3) $\textbf{cc}(\textbf{weaklySimplify}\,\alpha) \leq_{cc} \textbf{cc}\,\beta$;

(4) $\textbf{size}(\textbf{weaklySimplify}\,\alpha) \leq \textbf{size}\,\alpha$;

(5) $\textbf{numSyms}(\textbf{weaklySimplify}\,\alpha) \leq \textbf{numSyms}\,\alpha$; *and*

(6) $\textbf{numConcats}(\textbf{weaklySimplify}\,\alpha) \leq \textbf{numConcats}\,\alpha$.

Proof. By induction on regular expressions. □

Corollary 3.3.30

For all regular expressions α, $\textbf{weaklySimplify}\,\alpha \leq_{\textbf{simp}} \alpha$.

Weak Simplification

Proposition 3.3.31
*For all $\alpha \in$ **Reg**, if α is weakly simplified, then* **weaklySimplify** $\alpha = \alpha$.

Proof. By induction on regular expressions. □

Weak Simplification

Using our weak simplification algorithm, we can define an algorithm for calculating the language generated by a regular expression, when this language is finite, and for announcing that this language is infinite, otherwise.

Weak Simplification

Using our weak simplification algorithm, we can define an algorithm for calculating the language generated by a regular expression, when this language is finite, and for announcing that this language is infinite, otherwise.

First, we weakly simplify our regular expression, α, and call the resulting regular expression β. If β contains no closures, then we compute its meaning in the usual way. But, if β contains one or more closures, then its language will be infinite, and thus we can output a message saying that $L(\alpha)$ is infinite.

Weak Simplification in Forlan

The Forlan module Reg defines the following functions relating to weak simplification:

```
val weaklySimplified : reg -> bool
val weaklySimplify   : reg -> reg
val toStrSet         : reg -> str set
```

Weak Simplification in Forlan

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val weaklySimplified : reg -> bool
val weaklySimplify   : reg -> reg
val toStrSet         : reg -> str set
```

Here are some examples of how these functions can be used:

```
- val reg = Reg.input "";
@ (% + $0)(% + 00*0 + 0**)*
@ .
val reg = - : reg
- Reg.output("", Reg.weaklySimplify reg);
(% + 0* + 000*)*
val it = () : unit
- Reg.toStrSet reg;
language is infinite

uncaught exception Error
```

Weak Simplification in Forlan

```
- val reg' = Reg.input "";
@ (1 + %)(2 + $)(3 + %*)(4 + $*)
@ .
val reg' = - : reg
- StrSet.output("", Reg.toStrSet reg');
2, 12, 23, 24, 123, 124, 234, 1234
val it = () : unit
- Reg.output("", Reg.weaklySimplify reg');
(% + 1)2(% + 3)(% + 4)
val it = () : unit
- Reg.output
= ("",
=  Reg.weaklySimplify(Reg.fromString "(00*11*)*"));
(00*11*)*
val it = () : unit
```

Local and Global Simplification

In the book, we define a function/algorithm **hasEmp** $\in$ **Reg** $\rightarrow$ **Bool** such that, for all $\alpha \in$ **Reg**, % $\in L(\alpha)$ iff **hasEmp** $\alpha =$ **true**.

Local and Global Simplification

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In the book, we define a function/algorithm **obviousSubset** $\in$ **Reg** $\times$ **Reg** $\rightarrow$ **Bool** that is a *conservative approximation to subset testing*: for all $\alpha, \beta \in$ **Reg**, if **obviousSubset**$(\alpha, \beta) =$ **true**, then $L(\alpha) \subseteq L(\beta)$.

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On the positive side, we have that, e.g., **obviousSubset**$(0^*011^*1, 0^*1^*) =$ **true**.

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On the positive side, we have that, e.g., **obviousSubset**$(0^*011^*1, 0^*1^*) =$ **true**.

On the other hand, **obviousSubset**$((01)^*, (\% + 0)(10)^*(\% + 1)) =$ **false**, even though $L((01)^*) \subseteq L((\% + 0)(10)^*(\% + 1))$.

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On the positive side, we have that, e.g., **obviousSubset**$(0^*011^*1, 0^*1^*) =$ **true**.

On the other hand, **obviousSubset**$((01)^*, (\% + 0)(10)^*(\% + 1)) =$ **false**, even though $L((01)^*) \subseteq L((\% + 0)(10)^*(\% + 1))$.

In Section 3.13, we will learn of a less efficient algorithm that will provide a complete test for $L(\alpha) \subseteq L(\beta)$.

Simplification Rules

We have two kinds of simplification rules, which may be applied to subtrees of regular expressions:

- structural rules,
- reduction rules.

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Given some set of simplification rules and a regular expression α, when we generate the set of all regular expressions X that can be formed using these simplification rules starting from α, we add regular expressions to X in a series of *stages*. At stage 0, we have $\{\alpha\}$. At some stage n, we start with the regular expressions that we added at that stage (i.e., that were not added at any earlier stage). For each of these regular expressions, β,

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Structural Rules

There are nine *structural rules*, which preserve the alphabet, closure complexity, size, number of concatenations and number of symbols of a regular expression:

(1) $(\alpha + \beta) + \gamma \rightarrow \alpha + (\beta + \gamma)$.

(2) $\alpha + (\beta + \gamma) \rightarrow (\alpha + \beta) + \gamma$.

(3) $\alpha(\beta\gamma) \rightarrow (\alpha\beta)\gamma$.

(4) $(\alpha\beta)\gamma \rightarrow \alpha(\beta\gamma)$.

(5) $\alpha + \beta \rightarrow \beta + \alpha$.

(6) $\alpha^*\alpha \rightarrow \alpha\alpha^*$.

(7) $\alpha\alpha^* \rightarrow \alpha^*\alpha$.

(8) $\alpha(\beta\alpha)^* \rightarrow (\alpha\beta)^*\alpha$.

(9) $(\alpha\beta)^*\alpha \rightarrow \alpha(\beta\alpha)^*$.

Structural Rules

Because the structural rules preserve the size and alphabet of regular expressions, if we start with a regular expression α, there are only finitely many regular expressions that we can transform α into using structural rules.

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For even small regular expressions, there may be a very large number of ways to reorganize them using the structural rules. E.g., consider $\alpha_1 + \cdots + \alpha_n$, where $n \geq 1$ and $\alpha_1, \ldots, \alpha_n$ are distinct regular expressions. There are $n!$ ways of ordering the α_i.

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For even small regular expressions, there may be a very large number of ways to reorganize them using the structural rules. E.g., consider $\alpha_1 + \cdots + \alpha_n$, where $n \geq 1$ and $\alpha_1, \ldots, \alpha_n$ are distinct regular expressions. There are $n!$ ways of ordering the α_i. And there are $(2n)!/(n!)(n+1)!$ (these are the Catalan numbers) binary trees with exactly n leaves.

Structural Rules

Because the structural rules preserve the size and alphabet of regular expressions, if we start with a regular expression α, there are only finitely many regular expressions that we can transform α into using structural rules.

For even small regular expressions, there may be a very large number of ways to reorganize them using the structural rules. E.g., consider $\alpha_1 + \cdots + \alpha_n$, where $n \geq 1$ and $\alpha_1, \ldots, \alpha_n$ are distinct regular expressions. There are $n!$ ways of ordering the α_i. And there are $(2n)!/(n!)(n+1)!$ (these are the Catalan numbers) binary trees with exactly n leaves. Consequently, using structural rules (1), (2) and (5) (without making changes inside the α_i), we can reorganize $\alpha_1 + \cdots + \alpha_n$ into $(n!)(2n)!/(n!)(n+1)!$ regular expressions. For $n = 16$, this is about $7 * 10^{20}$.

Reduction Rules

There are 26 *reduction rules*, some of which make use of a conservative approximation ***sub*** to subset testing.

When $\alpha \rightarrow \beta$ because of a reduction rule, we have that **alphabet** $\beta \subseteq$ **alphabet** α and β **simp** α, where **simp** is the well-founded relation on **Reg** defined below.

Most of the rules strictly decrease a regular expression's closure complexity *and* size. The exceptions are labeled "cc" (for when the closure complexity strictly decreases, but the size strictly increases),

Reduction Rules

There are 26 *reduction rules*, some of which make use of a conservative approximation *sub* to subset testing.

When $\alpha \to \beta$ because of a reduction rule, we have that **alphabet** $\beta \subseteq$ **alphabet** α and β **simp** α, where **simp** is the well-founded relation on **Reg** defined below.

Most of the rules strictly decrease a regular expression's closure complexity *and* size. The exceptions are labeled "cc" (for when the closure complexity strictly decreases, but the size strictly increases), "concatenations" (for when the closure complexity and size are preserved, but the number of concatenations strictly decreases)

Reduction Rules

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When $\alpha \to \beta$ because of a reduction rule, we have that **alphabet** $\beta \subseteq$ **alphabet** α and β **simp** α, where **simp** is the well-founded relation on **Reg** defined below.

Most of the rules strictly decrease a regular expression's closure complexity *and* size. The exceptions are labeled "cc" (for when the closure complexity strictly decreases, but the size strictly increases), "concatenations" (for when the closure complexity and size are preserved, but the number of concatenations strictly decreases) or "symbols" (for when the closure complexity and size normally strictly decrease, but occasionally they and the number of concatenations stay they same, but the number of symbols strictly decreases).

Simplification Well-founded Relation

We define a relation **simp** on **Reg** by, for all $\alpha, \beta \in$ **Reg**, α **simp** β iff:

- **cc** $\alpha <_{cc}$ **cc** β; or
- **cc** $\alpha =$ **cc** β but **size** $\alpha <$ **size** β; or
- **cc** $\alpha =$ **cc** β and **size** $\alpha =$ **size** β, but **numConcats** $\alpha <$ **numConcats** β; or
- **cc** $\alpha =$ **cc** β, **size** $\alpha =$ **size** β and **numConcats** $\alpha =$ **numConcats** β, but **numSyms** $\alpha <$ **numSyms** β.

We have that **simp** $\subseteq <_{\textbf{simp}} \subseteq \leq_{\textbf{simp}}$.

Proposition 3.3.35
simp *is a well-founded relation on* **Reg**.

Proposition 3.3.36
simp *is transitive.*

Proposition 3.3.37
Suppose $\alpha, \beta, \gamma \in$ **Reg**.

(1) *If* α *and* β *have the same closure complexity, size, numbers of concatenations and numbers of symbols, and* β **simp** γ*, then* α **simp** γ.

(2) *If* α **simp** β*, and* β *and* γ *have the same closure complexity, size, numbers of concatenations and numbers of symbols, then* α **simp** γ.

(3) *If* $\alpha \leq_{\mathsf{simp}} \beta$ **simp** γ*, then* α **simp** γ.

(4) *If* α **simp** $\beta \leq_{\mathsf{simp}} \gamma$*, then* α **simp** γ.

Simplification Well-founded Relation

Proposition 3.3.38
Suppose $\alpha, \beta, \beta' \in \textbf{Reg}$, β' **simp** β, *pat* $\in$ **Path** *is valid for* α, *and* β *is the subtree of* α *at position pat. Let* α' *be the result of replacing the subtree at position pat in* α *by* β'*. Then* α' **simp** α*.*

Proof. By induction on α. □

Reduction Rules

(1) If *sub*(α, β), then $\alpha + \beta \to$

(2) $\alpha\beta_1 + \alpha\beta_2 \to$

(3) $\alpha_1\beta + \alpha_2\beta \to$

(4) If **hasEmp** α and *sub*(α, β^*), then $\alpha\beta^* \to$

(5) If **hasEmp** β and *sub*(β, α^*), then $\alpha^*\beta \to$

(6) If *sub*(α, β^*), then $(\alpha + \beta)^* \to$

(7) $(\alpha^* + \beta)^* \to$

(8) (concatenations) If **hasEmp** α and **hasEmp** β, then $(\alpha\beta)^* \to$

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(10) If **hasEmp** α and *sub*(α, β^*), then $(\alpha\beta)^* \to$

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(13) If **hasEmp** β and *sub*$(\beta, (\alpha+\gamma)^*)$, then
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(14) (cc) If **not(hasEmp** α**)** and $\mathbf{cc}\,\alpha \cup \overline{\mathbf{cc}\,\beta} <_{cc} \overline{\overline{\mathbf{cc}\,\beta}}$, then
$(\alpha\beta^*)^* \to$

(15) (cc) If **not(hasEmp** β**)** and $\overline{\mathbf{cc}\,\alpha} \cup \mathbf{cc}\,\beta <_{cc} \overline{\overline{\mathbf{cc}\,\alpha}}$, then
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Reduction Rules

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(15) (cc) If **not(hasEmp** β**)** and $\overline{\mathbf{cc}\,\alpha} \cup \mathbf{cc}\,\beta <_{cc} \overline{\overline{\mathbf{cc}\,\alpha}}$, then $(\alpha^*\beta)^* \to \% + (\alpha+\beta)^*\beta$.

(16) (cc) If **not(hasEmp** α**)** or **not(hasEmp** γ**)**, and $\mathbf{cc}\,\alpha \cup \overline{\mathbf{cc}\,\beta} \cup \mathbf{cc}\,\gamma <_{cc} \overline{\overline{\mathbf{cc}, \beta}}$, then $(\alpha\beta^*\gamma)^* \to \% + \alpha(\qquad)^*\gamma$.

(17) If *sub*$(\alpha\alpha^*, \beta)$, then $\alpha^* + \beta \to$

(18) If **hasEmp** β and *sub*$(\alpha\alpha\alpha^*, \beta)$, then $\alpha^* + \beta \to$

(19) (symbols) If $\alpha \notin \{\%, \$\}$ and *sub*(α^n, β), then $\alpha^{n+1}\alpha^* + \beta \to$

(12) If **hasEmp** α and *sub*$(\alpha, (\beta + \gamma)^*)$, then
$(\alpha\beta + \gamma)^* \rightarrow (\beta + \gamma)^*$.

(13) If **hasEmp** β and *sub*$(\beta, (\alpha + \gamma)^*)$, then
$(\alpha\beta + \gamma)^* \rightarrow (\alpha + \gamma)^*$.

(14) (cc) If **not(hasEmp** α**)** and $\mathbf{cc}\,\alpha \cup \overline{\mathbf{cc}\,\beta} <_{cc} \overline{\overline{\mathbf{cc}\,\beta}}$, then
$(\alpha\beta^*)^* \rightarrow \% + \alpha(\alpha + \beta)^*$.

(15) (cc) If **not(hasEmp** β**)** and $\overline{\mathbf{cc}\,\alpha} \cup \mathbf{cc}\,\beta <_{cc} \overline{\overline{\mathbf{cc}\,\alpha}}$, then
$(\alpha^*\beta)^* \rightarrow \% + (\alpha + \beta)^*\beta$.

(16) (cc) If **not(hasEmp** α**)** or **not(hasEmp** γ**)**, and
$\mathbf{cc}\,\alpha \cup \overline{\mathbf{cc}\,\beta} \cup \mathbf{cc}\,\gamma <_{cc} \overline{\overline{\mathbf{cc}, \beta}}$, then
$(\alpha\beta^*\gamma)^* \rightarrow \% + \alpha(\beta + \gamma\alpha)^*\gamma$.

(17) If *sub*$(\alpha\alpha^*, \beta)$, then $\alpha^* + \beta \rightarrow$

(18) If **hasEmp** β and *sub*$(\alpha\alpha\alpha^*, \beta)$, then $\alpha^* + \beta \rightarrow$

(19) (symbols) If $\alpha \notin \{\%, \$\}$ and *sub*(α^n, β), then
$\alpha^{n+1}\alpha^* + \beta \rightarrow$

Reduction Rules

(12) If **hasEmp** α and *sub*$(\alpha, (\beta+\gamma)^*)$, then $(\alpha\beta+\gamma)^* \to (\beta+\gamma)^*$.

(13) If **hasEmp** β and *sub*$(\beta, (\alpha+\gamma)^*)$, then $(\alpha\beta+\gamma)^* \to (\alpha+\gamma)^*$.

(14) (cc) If **not(hasEmp** α**)** and **cc** $\alpha \cup \overline{\textbf{cc}\,\beta} <_{cc} \overline{\overline{\textbf{cc}\,\beta}}$, then $(\alpha\beta^*)^* \to \% + \alpha(\alpha+\beta)^*$.

(15) (cc) If **not(hasEmp** β**)** and $\overline{\textbf{cc}\,\alpha} \cup$ **cc** $\beta <_{cc} \overline{\overline{\textbf{cc}\,\alpha}}$, then $(\alpha^*\beta)^* \to \% + (\alpha+\beta)^*\beta$.

(16) (cc) If **not(hasEmp** α**)** or **not(hasEmp** γ**)**, and **cc** $\alpha \cup \overline{\textbf{cc}\,\beta} \cup$ **cc** $\gamma <_{cc} \overline{\overline{\textbf{cc},\beta}}$, then $(\alpha\beta^*\gamma)^* \to \% + \alpha(\beta+\gamma\alpha)^*\gamma$.

(17) If *sub*$(\alpha\alpha^*, \beta)$, then $\alpha^* + \beta \to \% + \beta$.

(18) If **hasEmp** β and *sub*$(\alpha\alpha\alpha^*, \beta)$, then $\alpha^* + \beta \to$

(19) (symbols) If $\alpha \notin \{\%, \$\}$ and *sub*(α^n, β), then $\alpha^{n+1}\alpha^* + \beta \to$

Reduction Rules

(12) If **hasEmp** α and *sub*$(\alpha, (\beta+\gamma)^*)$, then $(\alpha\beta+\gamma)^* \to (\beta+\gamma)^*$.

(13) If **hasEmp** β and *sub*$(\beta, (\alpha+\gamma)^*)$, then $(\alpha\beta+\gamma)^* \to (\alpha+\gamma)^*$.

(14) (cc) If **not(hasEmp** α**)** and $\mathbf{cc}\,\alpha \cup \overline{\mathbf{cc}\,\beta} <_{cc} \overline{\overline{\mathbf{cc}\,\beta}}$, then $(\alpha\beta^*)^* \to \% + \alpha(\alpha+\beta)^*$.

(15) (cc) If **not(hasEmp** β**)** and $\overline{\mathbf{cc}\,\alpha} \cup \mathbf{cc}\,\beta <_{cc} \overline{\overline{\mathbf{cc}\,\alpha}}$, then $(\alpha^*\beta)^* \to \% + (\alpha+\beta)^*\beta$.

(16) (cc) If **not(hasEmp** α**)** or **not(hasEmp** γ**)**, and $\mathbf{cc}\,\alpha \cup \overline{\mathbf{cc}\,\beta} \cup \mathbf{cc}\,\gamma <_{cc} \overline{\overline{\mathbf{cc},\beta}}$, then $(\alpha\beta^*\gamma)^* \to \% + \alpha(\beta+\gamma\alpha)^*\gamma$.

(17) If *sub*$(\alpha\alpha^*, \beta)$, then $\alpha^* + \beta \to \% + \beta$.

(18) If **hasEmp** β and *sub*$(\alpha\alpha\alpha^*, \beta)$, then $\alpha^* + \beta \to \alpha + \beta$.

(19) (symbols) If $\alpha \notin \{\%, \$\}$ and *sub*(α^n, β), then $\alpha^{n+1}\alpha^* + \beta \to$

Reduction Rules

(12) If **hasEmp** α and *sub*$(\alpha, (\beta + \gamma)^*)$, then $(\alpha\beta + \gamma)^* \to (\beta + \gamma)^*$.

(13) If **hasEmp** β and *sub*$(\beta, (\alpha + \gamma)^*)$, then $(\alpha\beta + \gamma)^* \to (\alpha + \gamma)^*$.

(14) (cc) If **not(hasEmp** α**)** and $\mathbf{cc}\,\alpha \cup \overline{\mathbf{cc}\,\beta} <_{cc} \overline{\overline{\mathbf{cc}\,\beta}}$, then $(\alpha\beta^*)^* \to \% + \alpha(\alpha + \beta)^*$.

(15) (cc) If **not(hasEmp** β**)** and $\overline{\mathbf{cc}\,\alpha} \cup \mathbf{cc}\,\beta <_{cc} \overline{\overline{\mathbf{cc}\,\alpha}}$, then $(\alpha^*\beta)^* \to \% + (\alpha + \beta)^*\beta$.

(16) (cc) If **not(hasEmp** α**)** or **not(hasEmp** γ**)**, and $\mathbf{cc}\,\alpha \cup \overline{\mathbf{cc}\,\beta} \cup \mathbf{cc}\,\gamma <_{cc} \overline{\overline{\mathbf{cc}, \beta}}$, then $(\alpha\beta^*\gamma)^* \to \% + \alpha(\beta + \gamma\alpha)^*\gamma$.

(17) If *sub*$(\alpha\alpha^*, \beta)$, then $\alpha^* + \beta \to \% + \beta$.

(18) If **hasEmp** β and *sub*$(\alpha\alpha\alpha^*, \beta)$, then $\alpha^* + \beta \to \alpha + \beta$.

(19) (symbols) If $\alpha \notin \{\%, \$\}$ and *sub*(α^n, β), then $\alpha^{n+1}\alpha^* + \beta \to \alpha^n\alpha^* + \beta$.

Reduction Rules

(20) If $n \geq 2$, $l \geq 0$ and $2n - 1 < m_1 < \cdots < m_l$, then
$(\alpha^n + \alpha^{n+1} + \cdots + \alpha^{2n-1} + \alpha^{m_1} + \cdots + \alpha^{m_l})^* \to$

(21) (symbols) If $\alpha \notin \{\%, \$\}$, then $\alpha + \alpha\beta \to$

(22) (symbols) If $\alpha \notin \{\%, \$\}$, then $\alpha + \beta\alpha \to$

(23) $\alpha^*(\% + \beta(\alpha + \beta)^*) \to$

(24) $(\% + (\alpha + \beta)^*\alpha)\beta^* \to$

(25) If $sub(\alpha, \beta^*)$ and $sub(\beta, \alpha)$, then $\% + \alpha\beta^* \to$

(26) If $sub(\beta, \alpha^*)$ and $sub(\alpha, \beta)$, then $\% + \alpha^*\beta \to$

Reduction Rules

(20) If $n \geq 2$, $l \geq 0$ and $2n - 1 < m_1 < \cdots < m_l$, then $(\alpha^n + \alpha^{n+1} + \cdots + \alpha^{2n-1} + \alpha^{m_1} + \cdots + \alpha^{m_l})^* \to \% + \alpha^n\alpha^*$.

(21) (symbols) If $\alpha \notin \{\%, \$\}$, then $\alpha + \alpha\beta \to$

(22) (symbols) If $\alpha \notin \{\%, \$\}$, then $\alpha + \beta\alpha \to$

(23) $\alpha^*(\% + \beta(\alpha + \beta)^*) \to$

(24) $(\% + (\alpha + \beta)^*\alpha)\beta^* \to$

(25) If $sub(\alpha, \beta^*)$ and $sub(\beta, \alpha)$, then $\% + \alpha\beta^* \to$

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(21) (symbols) If $\alpha \notin \{\%, \$\}$, then $\alpha + \alpha\beta \to \alpha(\% + \beta)$.

(22) (symbols) If $\alpha \notin \{\%, \$\}$, then $\alpha + \beta\alpha \to$

(23) $\alpha^*(\% + \beta(\alpha + \beta)^*) \to$

(24) $(\% + (\alpha + \beta)^*\alpha)\beta^* \to$

(25) If $sub(\alpha, \beta^*)$ and $sub(\beta, \alpha)$, then $\% + \alpha\beta^* \to$

(26) If $sub(\beta, \alpha^*)$ and $sub(\alpha, \beta)$, then $\% + \alpha^*\beta \to$

Reduction Rules

(20) If $n \geq 2$, $l \geq 0$ and $2n-1 < m_1 < \cdots < m_l$, then $(\alpha^n + \alpha^{n+1} + \cdots + \alpha^{2n-1} + \alpha^{m_1} + \cdots + \alpha^{m_l})^* \to \% + \alpha^n\alpha^*$.

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(22) (symbols) If $\alpha \notin \{\%, \$\}$, then $\alpha + \beta\alpha \to (\% + \beta)\alpha$.

(23) $\alpha^*(\% + \beta(\alpha + \beta)^*) \to$

(24) $(\% + (\alpha + \beta)^*\alpha)\beta^* \to$

(25) If $sub(\alpha, \beta^*)$ and $sub(\beta, \alpha)$, then $\% + \alpha\beta^* \to$

(26) If $sub(\beta, \alpha^*)$ and $sub(\alpha, \beta)$, then $\% + \alpha^*\beta \to$

Reduction Rules

(20) If $n \geq 2$, $l \geq 0$ and $2n - 1 < m_1 < \cdots < m_l$, then $(\alpha^n + \alpha^{n+1} + \cdots + \alpha^{2n-1} + \alpha^{m_1} + \cdots + \alpha^{m_l})^* \to \% + \alpha^n\alpha^*$.

(21) (symbols) If $\alpha \notin \{\%, \$\}$, then $\alpha + \alpha\beta \to \alpha(\% + \beta)$.

(22) (symbols) If $\alpha \notin \{\%, \$\}$, then $\alpha + \beta\alpha \to (\% + \beta)\alpha$.

(23) $\alpha^*(\% + \beta(\alpha + \beta)^*) \to (\alpha + \beta)^*$.

(24) $(\% + (\alpha + \beta)^*\alpha)\beta^* \to$

(25) If $sub(\alpha, \beta^*)$ and $sub(\beta, \alpha)$, then $\% + \alpha\beta^* \to$

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Reduction Rules

(20) If $n \geq 2$, $l \geq 0$ and $2n - 1 < m_1 < \cdots < m_l$, then $(\alpha^n + \alpha^{n+1} + \cdots + \alpha^{2n-1} + \alpha^{m_1} + \cdots + \alpha^{m_l})^* \to \% + \alpha^n \alpha^*$.

(21) (symbols) If $\alpha \notin \{\%, \$\}$, then $\alpha + \alpha\beta \to \alpha(\% + \beta)$.

(22) (symbols) If $\alpha \notin \{\%, \$\}$, then $\alpha + \beta\alpha \to (\% + \beta)\alpha$.

(23) $\alpha^*(\% + \beta(\alpha + \beta)^*) \to (\alpha + \beta)^*$.

(24) $(\% + (\alpha + \beta)^*\alpha)\beta^* \to (\alpha + \beta)^*$.

(25) If $sub(\alpha, \beta^*)$ and $sub(\beta, \alpha)$, then $\% + \alpha\beta^* \to$

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Reduction Rules

(20) If $n \geq 2$, $l \geq 0$ and $2n-1 < m_1 < \cdots < m_l$, then
$(\alpha^n + \alpha^{n+1} + \cdots + \alpha^{2n-1} + \alpha^{m_1} + \cdots + \alpha^{m_l})^* \to \% + \alpha^n\alpha^*$.

(21) (symbols) If $\alpha \notin \{\%, \$\}$, then $\alpha + \alpha\beta \to \alpha(\% + \beta)$.

(22) (symbols) If $\alpha \notin \{\%, \$\}$, then $\alpha + \beta\alpha \to (\% + \beta)\alpha$.

(23) $\alpha^*(\% + \beta(\alpha + \beta)^*) \to (\alpha + \beta)^*$.

(24) $(\% + (\alpha + \beta)^*\alpha)\beta^* \to (\alpha + \beta)^*$.

(25) If $sub(\alpha, \beta^*)$ and $sub(\beta, \alpha)$, then $\% + \alpha\beta^* \to \beta^*$.

(26) If $sub(\beta, \alpha^*)$ and $sub(\alpha, \beta)$, then $\% + \alpha^*\beta \to$

Reduction Rules

(20) If $n \geq 2$, $l \geq 0$ and $2n - 1 < m_1 < \cdots < m_l$, then $(\alpha^n + \alpha^{n+1} + \cdots + \alpha^{2n-1} + \alpha^{m_1} + \cdots + \alpha^{m_l})^* \to \% + \alpha^n\alpha^*$.

(21) (symbols) If $\alpha \notin \{\%, \$\}$, then $\alpha + \alpha\beta \to \alpha(\% + \beta)$.

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(23) $\alpha^*(\% + \beta(\alpha + \beta)^*) \to (\alpha + \beta)^*$.

(24) $(\% + (\alpha + \beta)^*\alpha)\beta^* \to (\alpha + \beta)^*$.

(25) If $sub(\alpha, \beta^*)$ and $sub(\beta, \alpha)$, then $\% + \alpha\beta^* \to \beta^*$.

(26) If $sub(\beta, \alpha^*)$ and $sub(\alpha, \beta)$, then $\% + \alpha^*\beta \to \alpha^*$.

Local Simplification

Suppose *sub* is a conservative approximation to subset testing. We say that a regular expression α is *locally simplified with respect to sub*: iff

- α is weakly simplified, and
- α can't be transformed by our structural rules (which may be applied to subtrees) into a regular expression to which one of our reduction rules (which may be applied to subtrees) applies.

Local Simplification

The *local simplification of* a regular expression α *with respect to* a conservative approximation to subset testing ***sub*** proceeds as follows.

It calls its main function with the weak simplification, β, of α. Then $\beta \leq_{\mathsf{simp}} \alpha$, $\textbf{alphabet}\,\beta \subseteq \textbf{alphabet}\,\alpha$ and β is equivalent to α.

Local Simplification

The *local simplification of* a regular expression α *with respect to* a conservative approximation to subset testing ***sub*** proceeds as follows.

It calls its main function with the weak simplification, β, of α. Then $\beta \leq_{\mathsf{simp}} \alpha$, **alphabet** $\beta \subseteq$ **alphabet** α and β is equivalent to α.

The main function is defined by well-founded recursion on **simp**. When called with a weakly simplified α, it returns a β such that:

- β is locally simplified with respect to ***sub***;
- β is equivalent to α;
- **alphabet** $\beta \subseteq$ **alphabet** α; and
- $\beta \leq_{\mathsf{simp}} \alpha$.

Local Simplification

The main function works as follows:

- It generates the (finite) set X of all regular expressions **weaklySimplify** γ, such that α can be reorganized using the structural rules (allowing applications to subtrees) into a regular expression β, which can be transformed by a single application of one of our reduction rules (allowing applications to subtrees) into γ.
- If X is empty, then it returns α.
- Otherwise, it calls itself recursively on the simplest element, λ, of X (when X doesn't have a unique simplest element, the smallest of the simplest elements—in our total ordering on regular expressions—is selected).

Local Simplification

Because

- the structural rules (even applied to subtrees) preserve closure complexity, size, number of concatenations, and number of symbols,
- the reduction rules (even applied to subtrees) produce **simp**-predecessors, and
- and weak simplification respects $\leq_{\mathbf{simp}}$,

we have that λ **simp** α (and so $\lambda \leq_{\mathbf{simp}} \alpha$), so that the recursive call is legal. Furthermore, weak simplification, and all of the rules, either preserve or decrease (via $\subseteq$) the alphabet of regular expressions. Thus **alphabet** $\lambda \subseteq$ **alphabet** α. Finally, λ is equivalent to α, because all the rules and weak simplification preserve equivalence.

Local Simplification

We define a function/algorithm

$$\textbf{locallySimplify} \in (\textbf{Reg} \times \textbf{Reg} \to \textbf{Bool}) \to \textbf{Reg} \to \textbf{Reg}$$

by: for all conservative approximations to subset testing *sub*, and $\alpha \in \textbf{Reg}$, **locallySimplify** *sub* α is the result of running our local simplification algorithm on α, using *sub* as the conservative approximation to subset testing.

Theorem 3.3.39

For all conservative approximations to subset testing sub, and $\alpha \in \textbf{Reg}$*:*

- **locallySimplify** *sub* α *is locally simplified with respect to sub;*
- **locallySimplify** *sub* α *is equivalent to* α*;*
- **alphabet**(**locallySimplify** *sub* α) $\subseteq$ **alphabet** α*; and*
- **locallySimplify** *sub* $\alpha \leq_{\textbf{simp}} \alpha$*.*

Proof. By well-founded induction on **simp**. □

Local Simplification in Forlan

The Forlan module Reg provides the following functions relating to local simplification:

```
val locallySimplified    :
      (reg * reg -> bool) -> reg -> bool
val locallySimplify      :
      int option * (reg * reg -> bool) ->
      reg -> bool * reg
val locallySimplifyTrace :
      int option * (reg * reg -> bool) ->
      reg -> bool * reg
```

The argument of type `reg * reg -> bool` is a conservative approximation to subset testing. If the optional integer argument is `SOME` n, then at each recursive call of the principal function, only at most n structural reorganizations are considered. The returned boolean is `true` iff all the structural reorganizations of the returned regular expression were considered, and so it is locally simplified.

Local Simplification in Forlan

```
- val locSimped =
=   Reg.locallySimplified Reg.obviousSubset;
val locSimped = fn : reg -> bool
- locSimped(Reg.fromString "(1 + 00*1)*00*");
val it = false : bool
- locSimped(Reg.fromString "(0 + 1)*0");
val it = true : bool
- fun locSimp nOpt =
=       Reg.locallySimplify(nOpt, Reg.obviousSubset);
val locSimp = fn : int option -> reg -> bool * reg
- locSimp
= NONE
= (Reg.fromString "% + 0*0(0 + 1)* + 1*1(0 + 1)*");
val it = (true,-) : bool * reg
- Reg.output("", #2 it);
(0 + 1)*
val it = () : unit
```

Local Simplification in Forlan

```
- locSimp
= NONE
= (Reg.fromString "% + 1*0(0 + 1)* + 0*1(0 + 1)*");
val it = (true,-) : bool * reg
- Reg.output("", #2 it);
(0 + 1)*
val it = () : unit
- locSimp NONE (Reg.fromString "(1 + 00*1)*00*");
val it = (true,-) : bool * reg
- Reg.output("", #2 it);
(0 + 1)*0
val it = () : unit
```

Local Simplification in Forlan

```
- Reg.locallySimplifyTrace
= (NONE, Reg.obviousSubset)
= (Reg.fromString "0*(1 + 0*)*");
considered all 2 structural reorganizations of
0*(1 + 0*)*
0*(1 + 0*)* transformed by structural rule 5 at
position [2, 1] to 0*(0* + 1)* transformed by
reduction rule 7 at position [2] to 0*(0 + 1)*
considered all 2 structural reorganizations of
0*(0 + 1)*
0*(0 + 1)* transformed by reduction rule 4 at position
[] to (0 + 1)*
considered all 2 structural reorganizations of
(0 + 1)*
(0 + 1)* is locally simplified
val it = (true,-) : bool * reg
```

```
- val reg = Reg.input "";
@ 1 + (% + 0 + 2)(% + 0 + 2)*1 +
@ (1 + (% + 0 + 2)(% + 0 + 2)*1)
@ (% + 0 + 2 + 1(% + 0 + 2)*1)
@ (% + 0 + 2 + 1(% + 0 + 2)*1)*
@ .
val reg = - : reg
- Reg.equal(Reg.weaklySimplify reg, reg);
val it = true : bool
- val (b', reg') = locSimp (SOME 10) reg;
val b' = false : bool
val reg' = - : reg
- Reg.output("", reg');
(0 + 2)*1(0 + 2 + 1(0 + 2)*1)*
val it = () : unit
```

Local Simplification in Forlan

```
- val (b'', reg'') = locSimp (SOME 1000) reg';
val b'' = true : bool
val reg'' = - : reg
- Reg.output("", reg'');
(0 + 2)*1(0 + 2 + 1(0 + 2)*1)*
val it = () : unit
```

Global Simplification

Given a conservative approximation to subset testing *sub*, and a regular expression α, we say that α is *globally simplified with respect to sub* iff no strictly simpler regular expression can be found by an arbitrary number of applications (to subtrees) of weak simplification, structural rules and reduction rules.

Global Simplification

Given a conservative approximation to subset testing ***sub***, and a regular expression α, we say that α is *globally simplified with respect to **sub*** iff no strictly simpler regular expression can be found by an arbitrary number of applications (to subtrees) of weak simplification, structural rules and reduction rules.

The *global simplification of* a regular expression α *with respect to* a conservative approximation to subset testing ***sub*** consists of generating the set X of all regular expressions β that can formed from α by an arbitrary number of applications of weak simplification, the structural rules and the reduction rules (which may be applied to subtrees). All of the elements of X will have the same meaning as α, and will have alphabets that are subsets of the alphabet of α.

Global Simplification

Because

- weak simplification (even applied to subtrees) either preserves the closure complexity, size, numbers of concatenations and numbers of symbols of a regular expression, or results in a regular expression that is a **simp**-predecessor,
- the structural rules (even applied to subtrees) preserve the closure complexity, size, numbers of concatenations and numbers of symbols of a regular expression, and
- the reduction rules (even applied to subtrees) produce regular expressions that are **simp**-predecessors,

all the elements of X either are **simp**-predecessors of α or have the same closure complexity, size, numbers of concatenations and numbers of symbols as α.

Global Simplification

Because

- weak simplification (even applied to subtrees) either preserves the closure complexity, size, numbers of concatenations and numbers of symbols of a regular expression, or results in a regular expression that is a **simp**-predecessor,
- the structural rules (even applied to subtrees) preserve the closure complexity, size, numbers of concatenations and numbers of symbols of a regular expression, and
- the reduction rules (even applied to subtrees) produce regular expressions that are **simp**-predecessors,

all the elements of X either are **simp**-predecessors of α or have the same closure complexity, size, numbers of concatenations and numbers of symbols as α.

The book uses graph theory's König's lemma (every infinite finitely splitting tree has an infinite branch) to prove that the generation of X terminates.

Global Simplification

The simplest element of X is then selected (when there isn't a unique simplest element, the smallest of the simplest elements—in our total ordering on regular expressions—is selected). If this element is a **simp**-predecessor of α, it will be $\leq_{\mathbf{simp}} \alpha$. Otherwise, it will have the same closure complexity, size, numbers of concatenations and numbers of symbols as α. And it will be standardized, as weak simplification of a non-standardized regular expression will standardize it, making it more simplified. Thus it will be $\leq_{\mathbf{simp}} \alpha$. Similarly, it will be globally simplified with respect to *sub*, as otherwise it wouldn't be the simplest element of X.

Global Simplification

We define a function/algorithm

$$\textbf{globallySimplify} \in (\textbf{Reg} \times \textbf{Reg} \to \textbf{Bool}) \to \textbf{Reg} \to \textbf{Reg}$$

by: for all conservative approximations to subset testing *sub*, and $\alpha \in$ **Reg**, **globallySimplify** *sub* α is the result of running our global simplification algorithm on α, using *sub* as our conservative approximation to subset testing.

Theorem 3.3.42

For all conservative approximations to subset testing sub, and $\alpha \in$ **Reg**:

- **globallySimplify** *sub* α *is globally simplified with respect to sub;*
- **globallySimplify** *sub* α *is equivalent to* α*;*
- **alphabet**(**globallySimplify** *sub* α) $\subseteq$ **alphabet** α*; and*
- **globallySimplify** *sub* $\alpha \leq_{\textbf{simp}} \alpha$.

Global Simplification in Forlan

The Forlan module Reg provides the following functions relating to global simplification:

```
val globallySimplified    :
      (reg * reg -> bool) -> reg -> bool
val globallySimplifyTrace :
      int option * (reg * reg -> bool) ->
      reg -> bool * reg
val globallySimplify      :
      int option * (reg * reg -> bool) ->
      reg -> bool * reg
```

The argument of type `reg * reg -> bool` is a conservative approximation to subset testing. If the optional integer argument is `SOME` *n*, at most *n* candidates will be considered. The returned boolean is `true` iff all candidates were considered, and so the returned regular expression is globally simplified.

Global Simplification in Forlan

```
- fun globSimp nOpt =
=       Reg.globallySimplify
=       (nOpt, Reg.obviousSubset);
val globSimp = fn : int option -> reg -> bool * reg
- fun globSimpTr nOpt =
=       Reg.globallySimplifyTrace
=       (nOpt, Reg.obviousSubset);
val globSimpTr = fn : int option -> reg -> bool * reg
```

Global Simplification in Forlan

```
- globSimpTr NONE (Reg.fromString "(0*0)*");
considering candidates with explanations of length 0
simplest result now: (0*0)*
considering candidates with explanations of length 1
simplest result now: (0*0)* weakly simplifies to
(00*)*
simplest result now: (0*0)* transformed by reduction
rule 10 at position [] to 0*
considering candidates with explanations of length 2
considering candidates with explanations of length 3
considering candidates with explanations of length 4
considering candidates with explanations of length 5
considering candidates with explanations of length 6
search completed after considering 17 candidates with
maximum size 8
(0*0)* transformed by reduction rule 10 at position []
to 0* is globally simplified
val it = (true,-) : bool * reg
```

Global Simplification in Forlan

```
- locSimp NONE (Reg.fromString "(00*11*)*");
val it = (true,-) : bool * reg
- Reg.output("", #2 it);
% + 00*1(% + (0 + 1)*1)
val it = () : unit
- globSimp NONE (Reg.fromString "(00*11*)*");
val it = (true,-) : bool * reg
- Reg.output("", #2 it);
% + 0(0 + 1)*1
val it = () : unit
```