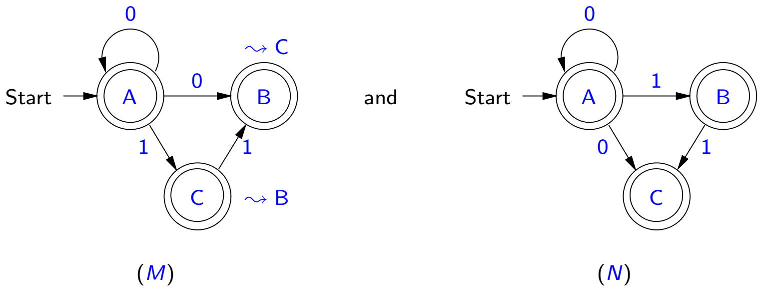
3.5: Isomorphism of Finite Automata

Let M and N be the finite automata



How are M and N related? Although they are not equal, they do have the same "structure", in that M can be turned into N by replacing A, B and C by A, C and B, respectively. When FAs have the same structure, we will say they are "isomorphic".

Definition of Isomorphism

An *isomorphism* h from an FA M to an FA N is a bijection from Q_M to Q_N such that

- $h\, s_M = s_N$;
- $\{\, h\, q \mid q \in A_M \,\} = A_N$; and
- $\{\, (h\, q), x \to (h\, r) \mid q, x \to r \in T_M \,\} = T_N$.

We define a relation **iso** on **FA** by: M **iso** N iff there is an isomorphism from M to N. We say that M and N are *isomorphic* iff M **iso** N.

Consider our example FAs M and N, and let h be the function

$$\{(\mathsf{A}, \mathsf{A}), (\mathsf{B}, \mathsf{C}), (\mathsf{C}, \mathsf{B})\}.$$

Then h is an isomorphism from M to N. Hence M **iso** N.

Properties of Isomorphism

Clearly, if M and N are isomorphic, then they have the same alphabet.

Proposition 3.5.1

The relation **iso** *is reflexive on* **FA**, *symmetric and transitive.*

Properties of Isomorphism

Proposition 3.5.2
Suppose M and N are isomorphic FAs. Then $L(M) \subseteq L(N)$.

Proof. Let h be an isomorphism from M to N. Suppose $w \in L(M)$. Then, there is a labeled path

$$lp = q_1 \overset{x_1}{\Rightarrow} q_2 \overset{x_2}{\Rightarrow} \cdots q_n \overset{x_n}{\Rightarrow} q_{n+1},$$

such that $w = x_1 x_2 \cdots x_n$, lp is valid for M, $q_1 = s_M$ and $q_{n+1} \in A_M$. Let

$$lp' = h\, q_1 \overset{x_1}{\Rightarrow} h\, q_2 \overset{x_2}{\Rightarrow} \cdots h\, q_n \overset{x_n}{\Rightarrow} h\, q_{n+1}.$$

Then the label of lp' is w, lp' is valid for N, $h\, q_1 = h\, s_M = s_N$ and $h\, q_{n+1} \in A_N$, showing that $w \in L(N)$. □

Proposition 3.5.3
Suppose M and N are isomorphic FAs. Then $M \approx N$.

Proof. Since M **iso** N, we have that N **iso** M, by Proposition 3.5.1. Thus, by Proposition 3.5.2, we have that $L(M) \subseteq L(N) \subseteq L(M)$. Hence $L(M) = L(N)$, i.e., $M \approx N$. □

Renaming States

The function **renameStates** takes in a pair (M, f), where $M \in$ **FA** and f is a bijection from Q_M to some set of symbols, and returns the **FA** produced from M by renaming M's states using the bijection f.

Proposition 3.5.4

Suppose M is an FA and f is a bijection from Q_M to some set of symbols. Then **renameStates**(M, f) **iso** M.

The following function is a special case of **renameStates**. The function **renameStatesCanonically** $\in$ **FA** $\rightarrow$ **FA** renames the states of an FA M to:

- A, B, etc., when the automaton has no more than 26 states (the smallest state of M will be renamed to A, the next smallest one to B, etc.); or
- $\langle 1 \rangle$, $\langle 2 \rangle$, etc., otherwise.

An Algorithm for Finding Isomorphisms

The book presents and proves the correctness of a relatively simple algorithm for finding an isomorphism from one FA to another, if one exists, and for indicating that there are no such isomorphisms, otherwise.

Isomorphism Finding/Checking in Forlan

The Forlan module `FA` also defines the functions

```
val isomorphism               : fa * fa * sym_rel -> bool
val findIsomorphism           : fa * fa -> sym_rel
val isomorphic                : fa * fa -> bool
val renameStates              : fa * sym_rel -> fa
val renameStatesCanonically : fa -> fa
```

Forlan Examples

Suppose that `fa1` and `fa2` have been bound to our example finite automata M and N, respectively. Then, here are some example uses of the above functions:

```
- val rel = FA.findIsomorphism(fa1, fa2);
val rel = - : sym_rel
- SymRel.output("", rel);
(A, A), (B, C), (C, B)
val it = () : unit
- FA.isomorphism(fa1, fa2, rel);
val it = true : bool
- FA.isomorphic(fa1, fa2);
val it = true : bool
```

Forlan Examples

```
- val rel' = FA.findIsomorphism(fa1, fa1);
val rel' = - : sym_rel
- SymRel.output("", rel');
(A, A), (B, B), (C, C)
val it = () : unit
- FA.isomorphism(fa1, fa1, rel');
val it = true : bool
- FA.isomorphism(fa1, fa2, rel');
val it = false : bool
```

Forlan Examples

```
- val rel'' = SymRel.input "";
@ (A, 2), (B, 1), (C, 0)
@ .
val rel'' = - : sym_rel
- val fa3 = FA.renameStates(fa1, rel'');
val fa3 = - : fa
- FA.output("", fa3);
{states} 0, 1, 2 {start state} 2
{accepting states} 0, 1, 2
{transitions} 0, 1 -> 1; 2, 0 -> 1 | 2; 2, 1 -> 0
val it = () : unit
```

Forlan Examples

```
- val fa4 = FA.renameStatesCanonically fa3;
val fa4 = - : fa
- FA.output("", fa4);
{states} A, B, C {start state} C
{accepting states} A, B, C
{transitions} A, 1 -> B; C, 0 -> B | C; C, 1 -> A
val it = () : unit
- FA.equal(fa4, fa1);
val it = false : bool
- FA.isomorphic(fa4, fa1);
val it = true : bool
```