3.6: Checking Acceptance and Finding Accepting Paths

In this section we study algorithms for:

- checking whether a string is accepted by a finite automaton; and
- finding a labeled path that explains why a string is accepted by a finite automaton.

Processing a String from a Set of States

Suppose M is a finite automaton. We define a function $\Delta_M \in \mathcal{P}\, Q_M \times \textbf{Str} \to \mathcal{P}\, Q_M$ by: $\Delta_M(P, w)$ is the set of all $r \in Q_M$ such that there is an $lp \in \textbf{LP}$ such that

- w is the label of lp;
- lp is valid for M;
- the start state of lp is in P; and
- r is the end state of lp.

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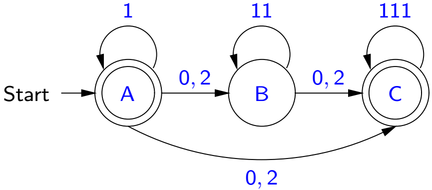
When the FA M is clear from the context, we sometimes abbreviate Δ_M to Δ.

Δ *Function Examples*

Suppose M is the finite automaton



Then, $\Delta_M(\{\mathsf{A}\}, 12111111) =$

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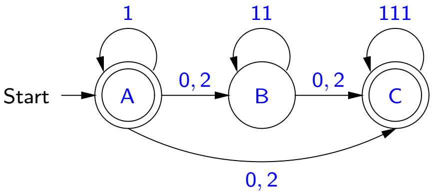

Then, $\Delta_M(\{A\}, 12111111) = \{B, C\}$, since

$$A \overset{1}{\Rightarrow} A \overset{2}{\Rightarrow} B \overset{11}{\Rightarrow} B \overset{11}{\Rightarrow} B \overset{11}{\Rightarrow} B \quad \text{and} \quad A \overset{1}{\Rightarrow} A \overset{2}{\Rightarrow} C \overset{111}{\Rightarrow} C \overset{111}{\Rightarrow} C$$

are all of the labeled paths that are labeled by 12111111, valid for M and whose start states are A.

Δ *Function Examples*

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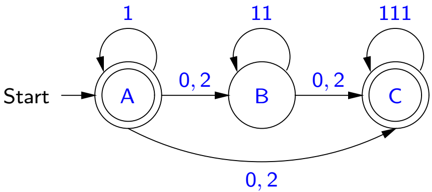

Then, $\Delta_M(\{A, B, C\}, 11) =$

Δ *Function Examples*

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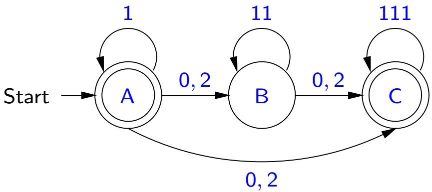

Then, $\Delta_M(\{A, B, C\}, 11) = \{A, B\}$, since

$$A \overset{1}{\Rightarrow} A \overset{1}{\Rightarrow} A \quad \text{and} \quad B \overset{11}{\Rightarrow} B$$

are all of the labeled paths that are labeled by 11 and valid for M.

An Algorithm for Calculating $\Delta(P, w)$

Suppose M is a finite automaton, $P \subseteq Q_M$ and $w \in \mathbf{Str}$. We can calculate $\Delta_M(P, w)$ as follows.

Let S be the set of all suffixes of w. Given $y \in S$, we write $\mathbf{pre}\, y$ for the unique x such that $w = xy$.

First, we generate the least subset X of $Q_M \times S$ such that:

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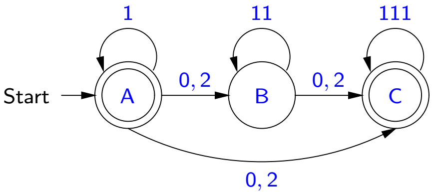
Calculating $\Delta(P, w)$

Suppose M is the finite automaton



Here are the elements of X, when $P = \{\mathsf{A}\}$ and $w = 2111$:

Calculating $\Delta(P, w)$

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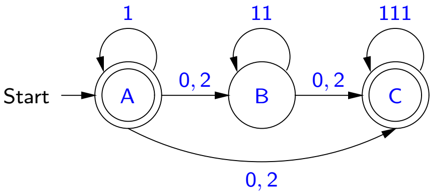

Here are the elements of X, when $P = \{\mathsf{A}\}$ and $w = 2111$:

- $(\mathsf{A}, 2111)$;

Calculating $\Delta(P, w)$

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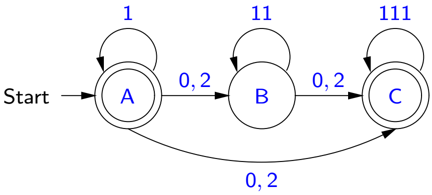

Here are the elements of X, when $P = \{A\}$ and $w = 2111$:

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- $(B, 111)$, because of the transition $A, 2 \rightarrow B$;

Calculating $\Delta(P, w)$

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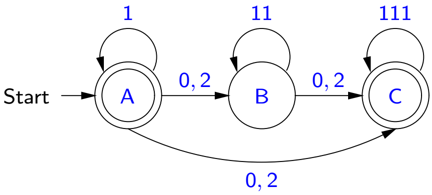

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Calculating $\Delta(P, w)$

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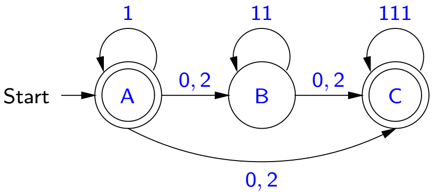

Here are the elements of X, when $P = \{A\}$ and $w = 2111$:

- $(A, 2111)$;
- $(B, 111)$, because of the transition $A, 2 \rightarrow B$;
- $(C, 111)$, because of the transition $A, 2 \rightarrow C$;
- $(B, 1)$, because of the transition $B, 11 \rightarrow B$;

Calculating $\Delta(P, w)$

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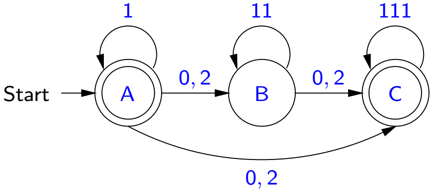

Here are the elements of X, when $P = \{\mathsf{A}\}$ and $w = 2111$:

- $(\mathsf{A}, 2111)$;
- $(\mathsf{B}, 111)$, because of the transition $\mathsf{A}, 2 \to \mathsf{B}$;
- $(\mathsf{C}, 111)$, because of the transition $\mathsf{A}, 2 \to \mathsf{C}$;
- $(\mathsf{B}, 1)$, because of the transition $\mathsf{B}, 11 \to \mathsf{B}$;
- $(\mathsf{C}, \%)$, because of the transition $\mathsf{C}, 111 \to \mathsf{C}$.

Lemma 3.6.1

For all $q \in Q_M$ *and* $y \in S$,

$$(q, y) \in X \quad \textit{iff}$$

Calculating $\Delta(P, w)$

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(1) Suppose $p \in P$ (so that $(p, w) \in X$). Then $p \in \Delta_M(P, \%)$. But $\mathbf{pre}\, w = \%$, so that $p \in \Delta_M(P, \mathbf{pre}\, w)$.

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(2) Suppose $q, r \in Q_M$, $x, y \in \textbf{Str}$, $(q, xy) \in X$ and $(q, x, r) \in T_M$. Assume the inductive hypothesis: $q \in \Delta_M(P, \textbf{pre}(xy))$.

showing that $r \in \Delta_M(P, \textbf{pre}\, y)$.

Calculating $\Delta(P, w)$

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Calculating $\Delta(P, w)$

Lemma 3.6.1
For all $q \in Q_M$ *and* $y \in S$,

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Proof. The "only if" (left-to-right) direction is by induction on X: we show that, for all $(q, y) \in X$, $q \in \Delta_M(P, \mathbf{pre}\, y)$.

(1) Suppose $p \in P$ (so that $(p, w) \in X$). Then $p \in \Delta_M(P, \%)$. But $\mathbf{pre}\, w = \%$, so that $p \in \Delta_M(P, \mathbf{pre}\, w)$.

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Calculating $\Delta(P, w)$

Proof (cont.). For the 'if" (right-to-left) direction, we have that there is a labeled path

$$q_1 \overset{x_1}{\Rightarrow} q_2 \overset{x_2}{\Rightarrow} \cdots q_n \overset{x_n}{\Rightarrow} q_{n+1},$$

that is valid for M and where $\textbf{pre}\, y = x_1 x_2 \cdots x_n$, $q_1 \in P$ and $q_{n+1} = q$. Since $q_1 \in P$ and $w = (\textbf{pre}\, y)y = x_1 x_2 \cdots x_n y$, we have that $(q_1, x_1 x_2 \cdots x_n y) = (q_1, w) \in X$, by (1). But $(q_1, x_1, q_2) \in T_M$, and thus $(q_2, x_2 \cdots x_n y) \in X$, by (2). Continuing on in this way (we could do this by mathematical induction), we finally get that $(q, y) = (q_{n+1}, y) \in X$. $\square$

Calculating $\Delta(P, w)$

Lemma 3.6.2
For all $q \in Q_M$, $(q, \%) \in X$ *iff* $q \in \Delta_M(P, w)$.

Proof. Suppose $(q, \%) \in X$. Lemma 3.6.1 tells us that $q \in \Delta_M(P, \mathbf{pre}\,\%)$. But $\mathbf{pre}\,\% = w$, and thus $q \in \Delta_M(P, w)$.

Suppose $q \in \Delta_M(P, w)$. Since $w = \mathbf{pre}\,\%$, we have that $q \in \Delta_M(P, \mathbf{pre}\,\%)$. Lemma 3.6.1 tells us that $(q, \%) \in X$. □

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By Lemma 3.6.2, we have that

$$\Delta_M(P, w) = \{\, q \in Q_M \mid (q, \%) \in X \,\}.$$

Thus, we return the set of all states q that are paired with $\%$ in X.

Checking String Acceptance and Finding Accepting Paths

Proposition 3.6.3

Suppose M is a finite automaton. Then

$$L(M) = \{\, w \in \mathbf{Str} \mid \Delta_M(\{s_M\}, w) \quad A_M \qquad \}.$$

Checking String Acceptance and Finding Accepting Paths

Proposition 3.6.3
Suppose M is a finite automaton. Then

$$L(M) = \{\, w \in \mathbf{Str} \mid \Delta_M(\{s_M\}, w) \cap A_M \neq \emptyset \,\}.$$

Finding Accepting Paths

Given a finite automaton M, subsets P, R of Q_M and a string w, how do we search for a labeled path that is labeled by w, valid for M, starts from an element of P, and ends with an element of R?

What we need to do is associate with each pair

$$(q, y)$$

of the set X that we generate when computing $\Delta_M(P, w)$ a labeled path lp such that lp is labeled by

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With a bit of care, we can ensure that these labeled paths are as short as possible.

As we generate the elements of X, we look for a pair of the form

Finding Accepting Paths

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of the set X that we generate when computing $\Delta_M(P, w)$ a labeled path lp such that lp is labeled by $\mathbf{pre}\, y$, lp is valid for M, the start state of lp is an element of P, and the end state of lp is q.

With a bit of care, we can ensure that these labeled paths are as short as possible.

As we generate the elements of X, we look for a pair of the form $(q, \%)$, where $q \in R$. Our answer will then be the labeled path associated with this pair.

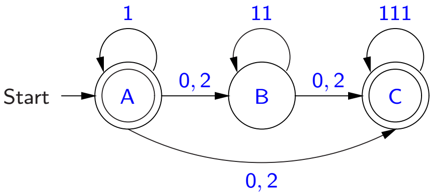
Checking Acceptance in Forlan

The Forlan module `FA` also contains the following functions for processing strings, checking string acceptance, and finding labeled paths:

```
val processStr       : fa -> sym set * str -> sym set
val accepted         : fa -> str -> bool
val findLP           : fa -> sym set * str * sym set -> lp
val findAcceptingLP : fa -> str -> lp
```

Forlan Examples

Suppose `fa` is the finite automaton



We begin by applying our four functions to `fa`, and giving names to the resulting functions:

```
- val processStr = FA.processStr fa;
val processStr = fn : sym set * str -> sym set
- val accepted = FA.accepted fa;
val accepted = fn : str -> bool
```

Continuing:

```
- val findLP = FA.findLP fa;
val findLP = fn : sym set * str * sym set -> lp
- val findAcceptingLP = FA.findAcceptingLP fa;
val findAcceptingLP = fn : str -> lp
```

Next, we'll define a set of states and a string to use later:

```
- val bs = SymSet.input "";
@ A, B, C
@ .
val bs = - : sym set
- val x = Str.input "";
@ 11
@ .
val x = [-,-] : str
```

Forlan Examples

Here are some example uses of our functions:

```
- SymSet.output("", processStr(bs, x));
A, B
val it = () : unit
- accepted(Str.input "");
@ 12111111
@ .
val it = true : bool
- accepted(Str.input "");
@ 1211
@ .
val it = false : bool
```

Forlan Examples

More examples:

```
- LP.output("", findLP(bs, x, bs));
B, 11 => B
val it = () : unit
- LP.output("", findAcceptingLP(Str.input ""));
@ 12111111
@ .
A, 1 => A, 2 => C, 111 => C, 111 => C
val it = () : unit
- LP.output("", findAcceptingLP(Str.input ""));
@ 222
@ .
no such labeled path exists

uncaught exception Error
```