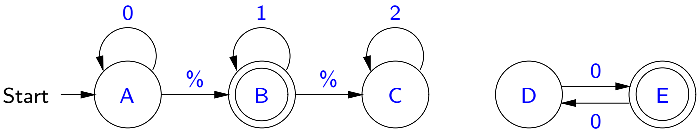
3.7: Simplification of Finite Automata

In this section, we: say what it means for a finite automaton to be simplified; study an algorithm for simplifying finite automata; and see how finite automata can be simplified in Forlan.

Suppose M is the finite automaton



What is odd about M?

First, there are no valid labeled paths from the start state to D and E, and so these states are redundant.

Second, there are no valid labeled paths from C to an accepting state, and so it is also redundant.

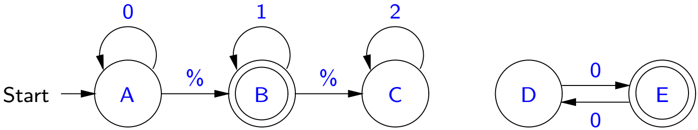
Useful States

Suppose M is a finite automaton. We say that a state $q \in Q_M$ is:

- *reachable in* M iff there is a labeled path lp such that lp is valid for M, the start state of lp is s_M, and the end state of lp is q;
- *live in* M iff there is a labeled path lp such that lp is valid for M, the start state of lp is q, and the end state of lp is in A_M;
- *dead in* M iff q is not live in M; and
- *useful in* M iff q is both reachable and live in M.

Useful States Example

Let M be our example finite automaton:



The reachable states of M are: A, B and C. The live states of M are: A, B, D and E. And, the useful states of M are: A and B.

Generating Reachable, Live and Useful States

There is a simple algorithm for generating the set of reachable states of a finite automaton M. We generate the least subset X of Q_M such that:

- $s_M \in X$; and
- for all $q, r \in Q_M$ and $x \in \mathbf{Str}$, if $q \in X$ and $(q, x, r) \in T_M$, then $r \in X$.

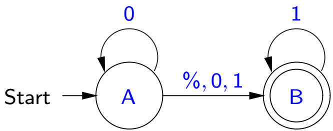
Similarly, there is a simple algorithm for generating the set of live states of a finite automaton M. We generate the least subset Y of Q_M such that:

- $A_M \subseteq Y$; and
- for all $q, r \in Q_M$ and $x \in \mathbf{Str}$, if $r \in Y$ and $(q, x, r) \in T_M$, then $q \in Y$.

Thus, we can generate the set of useful states of an FA by generating the set of reachable states, generating the set of live states, and intersecting those sets of states.

Redundant Transitions

Now, suppose N is the FA



What is odd about this machine?

Here, the transitions $(\mathsf{A}, 0, \mathsf{B})$ and $(\mathsf{A}, 1, \mathsf{B})$ are redundant.

Given an FA M and a finite subset U of $\{\, (q, x, r) \mid q, r \in Q_M$ and $x \in \textbf{Str}\,\}$, we write M/U for the FA that is identical to M except that its set of transitions is U.

If M is an FA and $(p, x, q) \in T_M$, we say that:

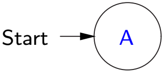
- (p, x, q) *is redundant in* M iff $q \in \Delta_N(\{p\}, x)$, where $N = M/(T_M - \{(p, x, q)\})$; and
- (p, x, q) *is irredundant in* M iff (p, x, q) is not redundant in M.

Definition of Simplification

We say that a finite automaton M is *simplified* iff either

- every state of M is useful, and every transition of M is irredundant; or
- $|Q_M| = 1$ and $A_M = T_M = \emptyset$.

Thus the FA



is simplified, even though its start state is not live, and is thus not useful.

Proposition 3.7.1
If M is a simplified finite automaton, then $\textbf{alphabet}(L(M)) = \textbf{alphabet}\, M$.

Algorithm for Removing Redundant Transitions

Given an FA M, $p, q \in Q_M$ and $x \in \mathbf{Str}$, we say that (p, x, q) *is implicit in* M iff $q \in \Delta_M(\{p\}, x)$.

Given an FA M, we define a function $\mathbf{remRedun}_M \in \mathcal{P}\, T_M \times \mathcal{P}\, T_M \to \mathcal{P}\, T_M$ by well-founded recursion on the size of its second argument.

For $U, V \subseteq T_M$, $\mathbf{remRedun}(U, V)$ proceeds as follows:

- If $V = \emptyset$, then it returns U.
- Otherwise, let v be the greatest element of $\{\, (q, x, r) \in V \mid$ there are no $q', r' \in \mathbf{Sym}$ and $y \in \mathbf{Str}$ such that $(q', y, r') \in V$ and $|y| > |x| \,\}$, and $V' = V - \{v\}$. If v is implicit in $M/(U \cup V')$, then $\mathbf{remRedun}$ returns the result of evaluating $\mathbf{remRedun}(U, V')$. Otherwise, it returns the result of evaluating $\mathbf{remRedun}(U \cup \{v\}, V')$.

Algorithm for Removing Redundant Transitions

In general, there are multiple—incompatible—ways of removing redundant transitions from an FA. **remRedun** is defined so as to favor removing transitions whose labels are longer; and among transitions whose labels have equal length, to favor removing transitions that are larger in our total ordering on transitions.

Simplification Algorithm

We define a function **simplify** $\in$ **FA** $\rightarrow$ **FA** by: **simplify** M is the finite automaton N produced by the following process.

- First, the useful states are M are determined.
- If s_M is not useful in M, the N is defined by:
  - $Q_N = \{s_M\}$;
  - $s_N = s_M$;
  - $A_N = \emptyset$; and
  - $T_N = \emptyset$.
- And, if s_M is useful in M, then N is **remRedun**$_{N'}(\emptyset, T_{N'})$, where N' is defined by
  - $Q_{N'} = \{\, q \in Q_M \mid q \text{ is useful in } M \,\}$;
  - $s_{N'} = s_M$;
  - $A_{N'} = A_M \cap Q_{N'} = \{\, q \in A_M \mid q \in Q_{N'} \,\}$; and
  - $T_{N'} = \{\, (q, x, r) \in T_M \mid q, r \in Q_{N'} \,\}$.

More on Simplification Algorithm

Proposition 3.7.3

Suppose M is a finite automaton. Then:

(1) $\textbf{simplify}\,M$ *is simplified;*

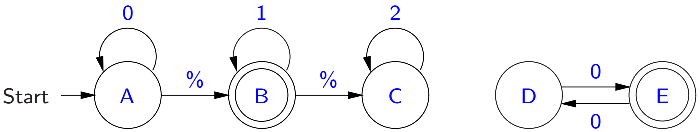
(2) $\textbf{simplify}\,M \approx M$;

(3) $Q_{\textbf{simplify}\,M} \subseteq Q_M$ *and* $T_{\textbf{simplify}\,M} \subseteq T_M$; *and*

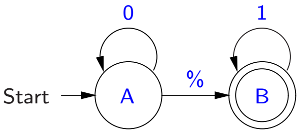
(4) $\textbf{alphabet}(\textbf{simplify}\,M) = \textbf{alphabet}(L(M)) \subseteq \textbf{alphabet}\,M$.

Simplification Examples

If M is the finite automaton

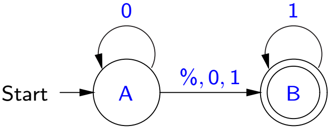


then **simplify** M is the finite automaton

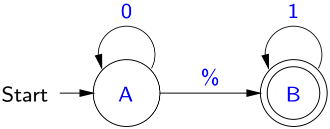


Simplification Examples

If N is the finite automaton



then **simplify** N is the finite automaton



Testing Whether $L(M) = \emptyset$

Our simplification algorithm gives us an algorithm for testing whether the language accepted by an FA M is empty. We first simplify M, calling the result N. We then test whether $A_N = \emptyset$. If the answer is "yes", clearly $L(M) = L(N) = \emptyset$. And if the answer is "no", then s_N is useful, and so N (and thus M) accepts at least one string.

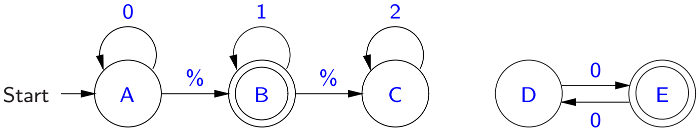
Simplification in Forlan

The Forlan module `FA` includes the following functions relating to the simplification of finite automata:

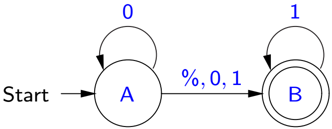
```
val simplify   : fa -> fa
val simplified : fa -> bool
```

Simplication Examples in Forlan

In the following, suppose `fa1` is the finite automaton

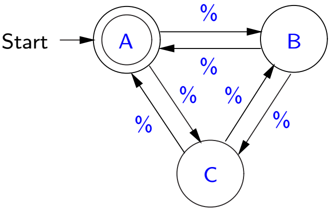


`fa2` is the finite automaton



Simplication Examples in Forlan

and `fa3` is the finita automaton



Simplication Examples in Forlan

Here are some example uses of `simplify` and `simplified`:

```
- FA.simplified fa1;
val it = false : bool
- val fa1' = FA.simplify fa1;
val fa1' = - : fa
- FA.output("", fa1');
{states} A, B {start state} A {accepting states} B
{transitions} A, % -> B; A, 0 -> A; B, 1 -> B
val it = () : unit
- FA.simplified fa1';
val it = true : bool
- val fa2' = FA.simplify fa2;
val fa2' = - : fa
- FA.output("", fa2');
{states} A, B {start state} A {accepting states} B
{transitions} A, % -> B; A, 0 -> A; B, 1 -> B
val it = () : unit
```

Simplication Examples in Forlan

```
- val fa3' = FA.simplify fa3;
val fa3' = - : fa
- FA.output("", fa3');
{states} A, B, C {start state} A {accepting states} A
{transitions} A, % -> B | C; B, % -> A; C, % -> A
val it = () : unit
```

Thus the simplification of fa3 resulted in the removal of the %-transitions between B and C.