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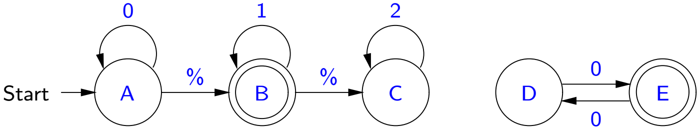

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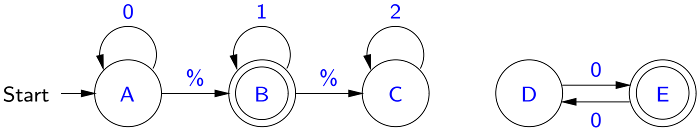

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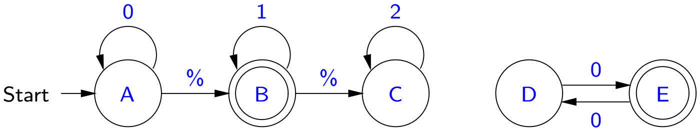

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First, there are no valid labeled paths from the start state to D and E, and so these states are redundant.

Second, there are no valid labeled paths from C to an accepting state, and so it is also redundant.

Useful States

Suppose M is a finite automaton. We say that a state $q \in Q_M$ is:

- *reachable in* M iff there is a labeled path lp such that lp is valid for M, the start state of lp is s_M, and the end state of lp is q;

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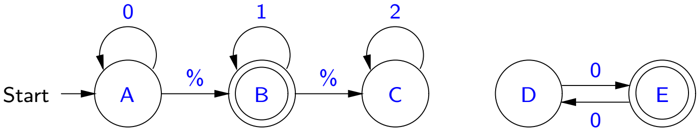
Let M be our example finite automaton:



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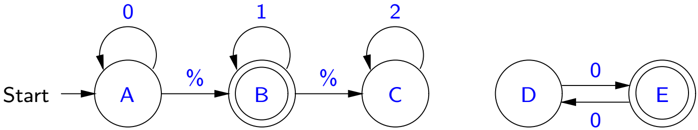

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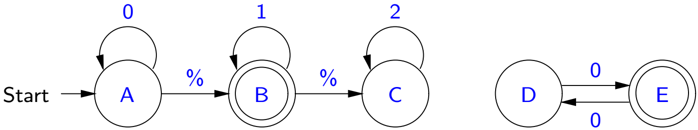

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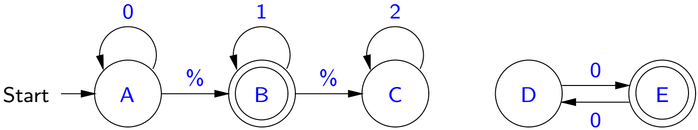

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Generating Reachable, Live and Useful States

There is a simple algorithm for generating the set of reachable states of a finite automaton M. We generate the least subset X of Q_M such that:

- $\in X$; and
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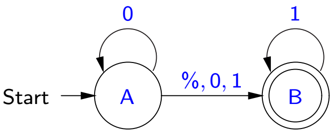
Thus, we can generate the set of useful states of an FA by generating the set of reachable states, generating the set of live states, and intersecting those sets of states.

Redundant Transitions

Now, suppose N is the FA



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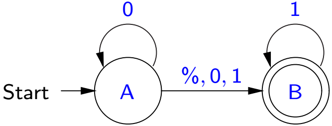

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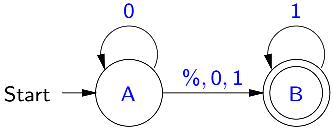

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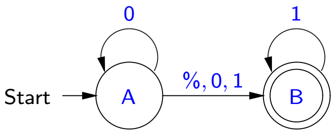

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If M is an FA and $(p, x, q) \in T_M$, we say that:

- (p, x, q) *is redundant in* M iff $q \in \Delta_N(\{p\}, x)$, where $N = M/(T_M - \{(p, x, q)\})$; and

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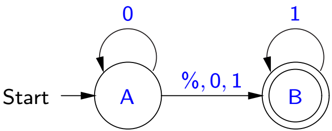

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- (p, x, q) *is irredundant in* M iff (p, x, q) is not redundant in M.

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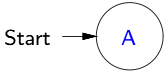

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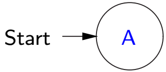

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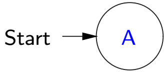

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Proposition 3.7.1
If M is a simplified finite automaton, then
$\textbf{alphabet}(L(M)) = \textbf{alphabet}\, M$.

Algorithm for Removing Redundant Transitions

Given an FA M, $p, q \in Q_M$ and $x \in \mathbf{Str}$, we say that (p, x, q) *is implicit in* M iff $q \in \Delta_M(\{p\}, x)$.

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- Otherwise, let v be the greatest element of $\{\, (q, x, r) \in V \mid$ there are no $q', r' \in \mathbf{Sym}$ and $y \in \mathbf{Str}$ such that $(q', y, r') \in V$ and $|y| > |x| \,\}$, and $V' = V - \{v\}$.

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Algorithm for Removing Redundant Transitions

In general, there are multiple—incompatible—ways of removing redundant transitions from an FA. **remRedun** is defined so as to favor removing transitions whose labels are longer; and among transitions whose labels have equal length, to favor removing transitions that are larger in our total ordering on transitions.

Simplification Algorithm

We define a function **simplify** $\in$ **FA** $\to$ **FA** by: **simplify** M is the finite automaton N produced by the following process.

- First, the useful states are M are determined.

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  - $Q_N = \{s_M\}$;
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  - $T_N = \emptyset$.
- And, if s_M is useful in M, then N is $\mathbf{remRedun}_{N'}(\emptyset, T_{N'})$, where N' is defined by
  - $Q_{N'} = \{\, q \in Q_M \mid q \text{ is useful in } M \,\}$;
  - $s_{N'} = s_M$;
  - $A_{N'} = A_M \cap Q_{N'} = \{\, q \in A_M \mid q \in Q_{N'} \,\}$; and
  - $T_{N'} = \{\, (q, x, r) \in T_M \mid q, r \in Q_{N'} \,\}$.

Proposition 3.7.3

Suppose M is a finite automaton. Then:

(1) $\textbf{simplify}\,M$ *is simplified;*

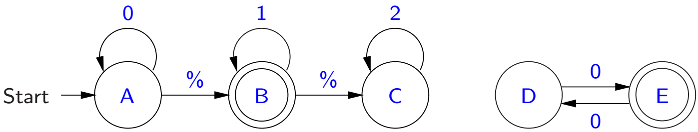
(2) $\textbf{simplify}\,M \approx M$;

(3) $Q_{\textbf{simplify}\,M} \subseteq Q_M$ *and* $T_{\textbf{simplify}\,M} \subseteq T_M$*; and*

(4) $\textbf{alphabet}(\textbf{simplify}\,M) = \textbf{alphabet}(L(M)) \subseteq \textbf{alphabet}\,M$.

Simplification Examples

If M is the finite automaton



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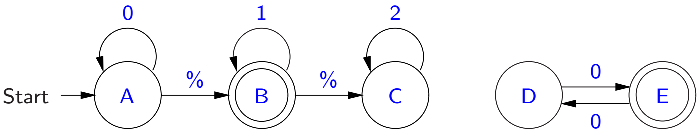

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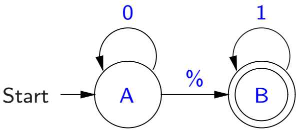

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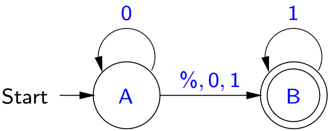

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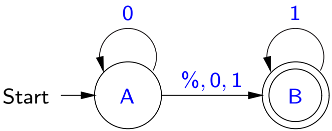

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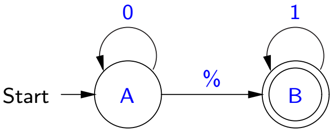


Testing Whether $L(M) = \emptyset$

Our simplification algorithm gives us an algorithm for testing whether the language accepted by an FA M is empty. We first simplify M, calling the result N. We then test whether

Testing Whether $L(M) = \emptyset$

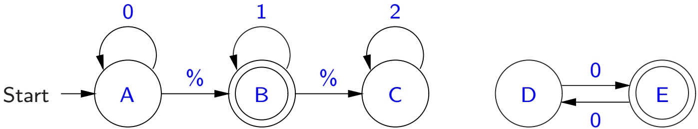
Our simplification algorithm gives us an algorithm for testing whether the language accepted by an FA M is empty. We first simplify M, calling the result N. We then test whether $A_N = \emptyset$. If the answer is "yes", clearly $L(M) = L(N) = \emptyset$. And if the answer is "no", then s_N is useful, and so N (and thus M) accepts at least one string.

Simplification in Forlan

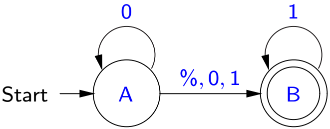
The Forlan module `FA` includes the following functions relating to the simplification of finite automata:

```
val simplify   : fa -> fa
val simplified : fa -> bool
```

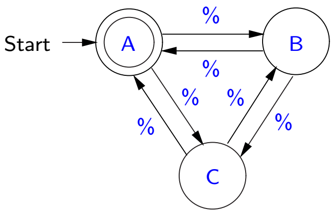
In the following, suppose `fa1` is the finite automaton



`fa2` is the finite automaton



and `fa3` is the finita automaton



Simplication Examples in Forlan

Here are some example uses of `simplify` and `simplified`:

```
- FA.simplified fa1;
val it = false : bool
- val fa1' = FA.simplify fa1;
val fa1' = - : fa
- FA.output("", fa1');
{states} A, B {start state} A {accepting states} B
{transitions} A, % -> B; A, 0 -> A; B, 1 -> B
val it = () : unit
- FA.simplified fa1';
val it = true : bool
- val fa2' = FA.simplify fa2;
val fa2' = - : fa
- FA.output("", fa2');
{states} A, B {start state} A {accepting states} B
{transitions} A, % -> B; A, 0 -> A; B, 1 -> B
val it = () : unit
```

Simplication Examples in Forlan

```
- val fa3' = FA.simplify fa3;
val fa3' = - : fa
- FA.output("", fa3');
{states} A, B, C {start state} A {accepting states} A
{transitions} A, % -> B | C; B, % -> A; C, % -> A
val it = () : unit
```

Thus the simplification of `fa3` resulted in the removal of the %-transitions between B and C.