Getting Started with Typed Functional Programming Using Standard ML

Alley Stoughton

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Standard ML

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- is strongly typed, featuring type inference

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  - data structures immutable, so sharing happens automatically

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  - data structures immutable, so sharing happens automatically
- has a powerful module language

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- is statically scoped
- uses eager evaluation (but lazy evaluation can be simulated)
- is mostly functional
  - imperative features, but downplayed
  - data structures immutable, so sharing happens automatically
- has a powerful module language
- has moderately good libraries

Compilers

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- MLton:

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  - whole program optimizing compiler

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There are two main compilers available:

- Standard ML of New Jersey (SML/NJ):
  - interactive front end
  - excellent support for separate compilation using the Compilation Manager (CM)
  - generates heap images, which can be loaded into executables
- MLton:
  - whole program optimizing compiler
  - generates executables
  - development normally done using SML/NJ

Examples

These slides and the code for my examples—plus links to more resources on Standard ML—are available on the web at:

`https://alleystoughton.us/getting-started-typed-fp-sml`

Using SML as a Calculator

```
-
```

Using SML as a Calculator

```
- 5 + 4;
```

Using SML as a Calculator

```
- 5 + 4;
val it = 9 : int
-
```

Using SML as a Calculator

```
- 5 + 4;
val it = 9 : int
- if 3 + it < 12
```

Using SML as a Calculator

```
- 5 + 4;
val it = 9 : int
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=
```

Using SML as a Calculator

```
- 5 + 4;
val it = 9 : int
- if 3 + it < 12
= then it mod 3
```

Using SML as a Calculator

```
- 5 + 4;
val it = 9 : int
- if 3 + it < 12
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=
```

Using SML as a Calculator

```
- 5 + 4;
val it = 9 : int
- if 3 + it < 12
= then it mod 3
= else it div 3;
```

Using SML as a Calculator

```
- 5 + 4;
val it = 9 : int
- if 3 + it < 12
= then it mod 3
= else it div 3;
val it = 3 : int
-
```

Using SML as a Calculator

```
- 5 + 4;
val it = 9 : int
- if 3 + it < 12
= then it mod 3
= else it div 3;
val it = 3 : int
- [1, 2] @ [3, 4];
```

Using SML as a Calculator

```
- 5 + 4;
val it = 9 : int
- if 3 + it < 12
= then it mod 3
= else it div 3;
val it = 3 : int
- [1, 2] @ [3, 4];
val it = [1,2,3,4] : int list
-
```

Using SML as a Calculator

```
- 5 + 4;
val it = 9 : int
- if 3 + it < 12
= then it mod 3
= else it div 3;
val it = 3 : int
- [1, 2] @ [3, 4];
val it = [1,2,3,4] : int list
- 0 :: it;
```

Using SML as a Calculator

```
- 5 + 4;
val it = 9 : int
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= then it mod 3
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val it = 3 : int
- [1, 2] @ [3, 4];
val it = [1,2,3,4] : int list
- 0 :: it;
val it = [0,1,2,3,4] : int list
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```
- 5 + 4;
val it = 9 : int
- if 3 + it < 12
= then it mod 3
= else it div 3;
val it = 3 : int
- [1, 2] @ [3, 4];
val it = [1,2,3,4] : int list
- 0 :: it;
val it = [0,1,2,3,4] : int list
- rev it;
```

Using SML as a Calculator

```
- 5 + 4;
val it = 9 : int
- if 3 + it < 12
= then it mod 3
= else it div 3;
val it = 3 : int
- [1, 2] @ [3, 4];
val it = [1,2,3,4] : int list
- 0 :: it;
val it = [0,1,2,3,4] : int list
- rev it;
val it = [4,3,2,1,0] : int list
-
```

Using SML as a Calculator

```
- 5 + 4;
val it = 9 : int
- if 3 + it < 12
= then it mod 3
= else it div 3;
val it = 3 : int
- [1, 2] @ [3, 4];
val it = [1,2,3,4] : int list
- 0 :: it;
val it = [0,1,2,3,4] : int list
- rev it;
val it = [4,3,2,1,0] : int list
- tl it;
```

Using SML as a Calculator

```
- 5 + 4;
val it = 9 : int
- if 3 + it < 12
= then it mod 3
= else it div 3;
val it = 3 : int
- [1, 2] @ [3, 4];
val it = [1,2,3,4] : int list
- 0 :: it;
val it = [0,1,2,3,4] : int list
- rev it;
val it = [4,3,2,1,0] : int list
- tl it;
val it = [3,2,1,0] : int list
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```

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- 0 :: it;
val it = [0,1,2,3,4] : int list
- rev it;
val it = [4,3,2,1,0] : int list
- tl it;
val it = [3,2,1,0] : int list
- hd it;
```

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```
- 5 + 4;
val it = 9 : int
- if 3 + it < 12
= then it mod 3
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- [1, 2] @ [3, 4];
val it = [1,2,3,4] : int list
- 0 :: it;
val it = [0,1,2,3,4] : int list
- rev it;
val it = [4,3,2,1,0] : int list
- tl it;
val it = [3,2,1,0] : int list
- hd it;
val it = 3 : int
```

Using SML as a Calculator

```
-
```

Using SML as a Calculator

```
- (4 * 9, 5 < 7);
```

Using SML as a Calculator

```
- (4 * 9, 5 < 7);
val it = (36,true) : int * bool
```

Declarations and Local Declarations

-

Declarations and Local Declarations

```
- val x = 4 + 8;
```

Declarations and Local Declarations

```
- val x = 4 + 8;
val x = 12 : int
-
```

Declarations and Local Declarations

```
- val x = 4 + 8;
val x = 12 : int
- val y = x * x;
```

Declarations and Local Declarations

```
- val x = 4 + 8;
val x = 12 : int
- val y = x * x;
val y = 144 : int
-
```

Declarations and Local Declarations

```
- val x = 4 + 8;
val x = 12 : int
- val y = x * x;
val y = 144 : int
- let val x = x + y
```

Declarations and Local Declarations

```
- val x = 4 + 8;
val x = 12 : int
- val y = x * x;
val y = 144 : int
- let val x = x + y
=
```

Declarations and Local Declarations

```
- val x = 4 + 8;
val x = 12 : int
- val y = x * x;
val y = 144 : int
- let val x = x + y
= in (x, 2 * x, 3 * x) end;
```

Declarations and Local Declarations

```
- val x = 4 + 8;
val x = 12 : int
- val y = x * x;
val y = 144 : int
- let val x = x + y
= in (x, 2 * x, 3 * x) end;
val it = (156,312,468) : int * int * int
-
```

Declarations and Local Declarations

```
- val x = 4 + 8;
val x = 12 : int
- val y = x * x;
val y = 144 : int
- let val x = x + y
= in (x, 2 * x, 3 * x) end;
val it = (156,312,468) : int * int * int
- #2 it;
```

Declarations and Local Declarations

```
- val x = 4 + 8;
val x = 12 : int
- val y = x * x;
val y = 144 : int
- let val x = x + y
= in (x, 2 * x, 3 * x) end;
val it = (156,312,468) : int * int * int
- #2 it;
val it = 312 : int
```

Function Definitions

-

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```
- fun fact n =
```

Function Definitions

```
- fun fact n =
=
```

Function Definitions

```
- fun fact n =
=       if n = 0
```

Function Definitions

```
- fun fact n =
=       if n = 0
=
```

Function Definitions

```
- fun fact n =
=       if n = 0
=       then 1
```

Function Definitions

```
- fun fact n =
=       if n = 0
=       then 1
=
```

Function Definitions

```
- fun fact n =
=       if n = 0
=       then 1
=       else n * fact(n - 1);
```

Function Definitions

```
- fun fact n =
=       if n = 0
=       then 1
=       else n * fact(n - 1);
val fact = fn : int -> int
-
```

Function Definitions

```
- fun fact n =
=       if n = 0
=       then 1
=       else n * fact(n - 1);
val fact = fn : int -> int
- fact 6;
```

Function Definitions

```
- fun fact n =
=       if n = 0
=       then 1
=       else n * fact(n - 1);
val fact = fn : int -> int
- fact 6;
val it = 720 : int
```

Function Definitions and Pattern Matching

-

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```
- fun fact n =
```

Function Definitions and Pattern Matching

```
- fun fact n =
=
```

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```
- fun fact n =
=       case n of
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Function Definitions and Pattern Matching

```
- fun fact n =
=       case n of
=            0 => 1
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Function Definitions and Pattern Matching

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- fun fact n =
=       case n of
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```

Function Definitions and Pattern Matching

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- fun fact n =
=       case n of
=            0 => 1
=          | n => n * fact(n - 1);
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Function Definitions and Pattern Matching

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- fun fact n =
=       case n of
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```

Function Definitions and Pattern Matching

```
- fun fact n =
=       case n of
=            0 => 1
=          | n => n * fact(n - 1);
val fact = fn : int -> int
- fact 7;
```

Function Definitions and Pattern Matching

```
- fun fact n =
=       case n of
=            0 => 1
=          | n => n * fact(n - 1);
val fact = fn : int -> int
- fact 7;
val it = 5040 : int
-
```

Function Definitions and Pattern Matching

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- fun fact n =
=       case n of
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val fact = fn : int -> int
- fact 7;
val it = 5040 : int
- fun fact 0 = 1
```

Function Definitions and Pattern Matching

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- fun fact n =
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val fact = fn : int -> int
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- fun fact 0 = 1
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Function Definitions and Pattern Matching

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- fun fact n =
=       case n of
=            0 => 1
=          | n => n * fact(n - 1);
val fact = fn : int -> int
- fact 7;
val it = 5040 : int
- fun fact 0 = 1
=   | fact n = n * fact(n - 1);
```

Function Definitions and Pattern Matching

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- fun fact n =
=       case n of
=            0 => 1
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- fun fact 0 = 1
=   | fact n = n * fact(n - 1);
val fact = fn : int -> int
-
```

Function Definitions and Pattern Matching

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- fun fact n =
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val fact = fn : int -> int
- fact 8;
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Function Definitions and Pattern Matching

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- fun fact n =
=       case n of
=            0 => 1
=          | n => n * fact(n - 1);
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- fact 7;
val it = 5040 : int
- fun fact 0 = 1
=   | fact n = n * fact(n - 1);
val fact = fn : int -> int
- fact 8;
val it = 40320 : int
```

Tail Recursion

-

Compilers for functional programming languages translate *tail recursion* into loops, not allocating stack frames.

Tail Recursion

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Tail Recursion

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- fun fact n =
=       let fun fct(0, m) = m
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- fun fact n =
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=       in fct(n, 1) end;
```

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=       in fct(n, 1) end;
val fact = fn : int -> int
- fact 6;
```

Compilers for functional programming languages translate *tail recursion* into loops, not allocating stack frames.

Tail Recursion

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- fun fact n =
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=       in fct(n, 1) end;
val fact = fn : int -> int
- fact 6;
val it = 720 : int
```

Compilers for functional programming languages translate *tail recursion* into loops, not allocating stack frames.

Polymorphism and List Processing Functions

-

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- fun rev xs =
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Polymorphism and List Processing Functions

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Polymorphism and List Processing Functions

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- fun rev xs =
=       if null xs
=       then nil
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Polymorphism and List Processing Functions

```
- fun rev xs =
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=       else rev(tl xs) @ [hd xs];
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Polymorphism and List Processing Functions

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- fun rev xs =
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```

Polymorphism and List Processing Functions

```
- fun rev xs =
=       if null xs
=       then nil
=       else rev(tl xs) @ [hd xs];
val rev = fn : 'a list -> 'a list
- rev[1, 3, 5, 7];
```

Polymorphism and List Processing Functions

```
- fun rev xs =
=       if null xs
=       then nil
=       else rev(tl xs) @ [hd xs];
val rev = fn : 'a list -> 'a list
- rev[1, 3, 5, 7];
val it = [7,5,3,1] : int list
-
```

Polymorphism and List Processing Functions

```
- fun rev xs =
=       if null xs
=       then nil
=       else rev(tl xs) @ [hd xs];
val rev = fn : 'a list -> 'a list
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val it = [7,5,3,1] : int list
- fun rev nil       = nil
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val rev = fn : 'a list -> 'a list
- rev[1, 3, 5, 7];
val it = [7,5,3,1] : int list
- fun rev nil       = nil
=   | rev (x :: xs) = rev xs @ [x];
```

Polymorphism and List Processing Functions

```
- fun rev xs =
=       if null xs
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List Processing Functions and Tail Recursion

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=
```

List Processing Functions and Tail Recursion

```
- fun rev xs =
=       let fun rv(nil,     ys) = ys
=             | rv(x :: xs, ys) = rv(xs, x :: ys)
```

List Processing Functions and Tail Recursion

```
- fun rev xs =
=       let fun rv(nil,     ys) = ys
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=
```

List Processing Functions and Tail Recursion

```
- fun rev xs =
=       let fun rv(nil,     ys) = ys
=             | rv(x :: xs, ys) = rv(xs, x :: ys)
=       in rv(xs, nil) end;
```

List Processing Functions and Tail Recursion

```
- fun rev xs =
=       let fun rv(nil,     ys) = ys
=             | rv(x :: xs, ys) = rv(xs, x :: ys)
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=       in rv(xs, nil) end;
val rev = fn : 'a list -> 'a list
- rev[1, 3, 5, 7];
val it = [7,5,3,1] : int list
```

Anonymous and Higher-order Functions

-

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```
- fn x => x + 1;
```

Anonymous and Higher-order Functions

```
- fn x => x + 1;
val it = fn : int -> int
-
```

Anonymous and Higher-order Functions

```
- fn x => x + 1;
val it = fn : int -> int
- it(3 + 4);
```

Anonymous and Higher-order Functions

```
- fn x => x + 1;
val it = fn : int -> int
- it(3 + 4);
val it = 8 : int
-
```

Anonymous and Higher-order Functions

```
- fn x => x + 1;
val it = fn : int -> int
- it(3 + 4);
val it = 8 : int
- map;
```

Anonymous and Higher-order Functions

```
- fn x => x + 1;
val it = fn : int -> int
- it(3 + 4);
val it = 8 : int
- map;
val it = fn : ('a -> 'b) -> 'a list -> 'b list
-
```

Anonymous and Higher-order Functions

```
- fn x => x + 1;
val it = fn : int -> int
- it(3 + 4);
val it = 8 : int
- map;
val it = fn : ('a -> 'b) -> 'a list -> 'b list
- map (fn x => x + 1) [1, 3, 5];
```

Anonymous and Higher-order Functions

```
- fn x => x + 1;
val it = fn : int -> int
- it(3 + 4);
val it = 8 : int
- map;
val it = fn : ('a -> 'b) -> 'a list -> 'b list
- map (fn x => x + 1) [1, 3, 5];
val it = [2,4,6] : int list
```

Anonymous and Higher-order Functions

-

Anonymous and Higher-order Functions

```
- List.exists;
```

Anonymous and Higher-order Functions

```
- List.exists;
[autoloading]
[autoloading done]
val it = fn : ('a -> bool) -> 'a list -> bool
-
```

Anonymous and Higher-order Functions

```
- List.exists;
[autoloading]
[autoloading done]
val it = fn : ('a -> bool) -> 'a list -> bool
- List.exists
```

Anonymous and Higher-order Functions

```
- List.exists;
[autoloading]
[autoloading done]
val it = fn : ('a -> bool) -> 'a list -> bool
- List.exists
=
```

Anonymous and Higher-order Functions

```
- List.exists;
[autoloading]
[autoloading done]
val it = fn : ('a -> bool) -> 'a list -> bool
- List.exists
= (fn x => x mod 2 = 0)
```

Anonymous and Higher-order Functions

```
- List.exists;
[autoloading]
[autoloading done]
val it = fn : ('a -> bool) -> 'a list -> bool
- List.exists
= (fn x => x mod 2 = 0)
=
```

Anonymous and Higher-order Functions

```
- List.exists;
[autoloading]
[autoloading done]
val it = fn : ('a -> bool) -> 'a list -> bool
- List.exists
= (fn x => x mod 2 = 0)
= [1, 2, 3, 4, 5, 6, 7];
```

Anonymous and Higher-order Functions

```
- List.exists;
[autoloading]
[autoloading done]
val it = fn : ('a -> bool) -> 'a list -> bool
- List.exists
= (fn x => x mod 2 = 0)
= [1, 2, 3, 4, 5, 6, 7];
val it = true : bool
-
```

Anonymous and Higher-order Functions

```
- List.exists;
[autoloading]
[autoloading done]
val it = fn : ('a -> bool) -> 'a list -> bool
- List.exists
= (fn x => x mod 2 = 0)
= [1, 2, 3, 4, 5, 6, 7];
val it = true : bool
- List.filter;
```

Anonymous and Higher-order Functions

```
- List.exists;
[autoloading]
[autoloading done]
val it = fn : ('a -> bool) -> 'a list -> bool
- List.exists
= (fn x => x mod 2 = 0)
= [1, 2, 3, 4, 5, 6, 7];
val it = true : bool
- List.filter;
val it = fn : ('a -> bool) -> 'a list -> 'a list
-
```

Anonymous and Higher-order Functions

```
- List.exists;
[autoloading]
[autoloading done]
val it = fn : ('a -> bool) -> 'a list -> bool
- List.exists
= (fn x => x mod 2 = 0)
= [1, 2, 3, 4, 5, 6, 7];
val it = true : bool
- List.filter;
val it = fn : ('a -> bool) -> 'a list -> 'a list
- List.filter
```

Anonymous and Higher-order Functions

```
- List.exists;
[autoloading]
[autoloading done]
val it = fn : ('a -> bool) -> 'a list -> bool
- List.exists
= (fn x => x mod 2 = 0)
= [1, 2, 3, 4, 5, 6, 7];
val it = true : bool
- List.filter;
val it = fn : ('a -> bool) -> 'a list -> 'a list
- List.filter
=
```

Anonymous and Higher-order Functions

```
- List.exists;
[autoloading]
[autoloading done]
val it = fn : ('a -> bool) -> 'a list -> bool
- List.exists
= (fn x => x mod 2 = 0)
= [1, 2, 3, 4, 5, 6, 7];
val it = true : bool
- List.filter;
val it = fn : ('a -> bool) -> 'a list -> 'a list
- List.filter
= (fn x => x mod 2 = 0)
```

Anonymous and Higher-order Functions

```
- List.exists;
[autoloading]
[autoloading done]
val it = fn : ('a -> bool) -> 'a list -> bool
- List.exists
= (fn x => x mod 2 = 0)
= [1, 2, 3, 4, 5, 6, 7];
val it = true : bool
- List.filter;
val it = fn : ('a -> bool) -> 'a list -> 'a list
- List.filter
= (fn x => x mod 2 = 0)
=
```

Anonymous and Higher-order Functions

```
- List.exists;
[autoloading]
[autoloading done]
val it = fn : ('a -> bool) -> 'a list -> bool
- List.exists
= (fn x => x mod 2 = 0)
= [1, 2, 3, 4, 5, 6, 7];
val it = true : bool
- List.filter;
val it = fn : ('a -> bool) -> 'a list -> 'a list
- List.filter
= (fn x => x mod 2 = 0)
= [1, 2, 3, 4, 5, 6, 7];
```

Anonymous and Higher-order Functions

```
- List.exists;
[autoloading]
[autoloading done]
val it = fn : ('a -> bool) -> 'a list -> bool
- List.exists
= (fn x => x mod 2 = 0)
= [1, 2, 3, 4, 5, 6, 7];
val it = true : bool
- List.filter;
val it = fn : ('a -> bool) -> 'a list -> 'a list
- List.filter
= (fn x => x mod 2 = 0)
= [1, 2, 3, 4, 5, 6, 7];
val it = [2,4,6] : int list
```

Option Types

-

Option Types

```
- NONE;
```

Option Types

```
- NONE;
val it = NONE : 'a option
-
```

Option Types

```
- NONE;
val it = NONE : 'a option
- SOME 5;
```

Option Types

```
- NONE;
val it = NONE : 'a option
- SOME 5;
val it = SOME 5 : int option
-
```

Option Types

```
- NONE;
val it = NONE : 'a option
- SOME 5;
val it = SOME 5 : int option
- SOME true;
```

Option Types

```
- NONE;
val it = NONE : 'a option
- SOME 5;
val it = SOME 5 : int option
- SOME true;
val it = SOME true : bool option
-
```

Option Types

```
- NONE;
val it = NONE : 'a option
- SOME 5;
val it = SOME 5 : int option
- SOME true;
val it = SOME true : bool option
- valOf(SOME false);
```

Option Types

```
- NONE;
val it = NONE : 'a option
- SOME 5;
val it = SOME 5 : int option
- SOME true;
val it = SOME true : bool option
- valOf(SOME false);
val it = false : bool
```

Option Types

-

Option Types

```
- fun firstPos(f, ys) =
```

Option Types

```
- fun firstPos(f, ys) =
=
```

Option Types

```
- fun firstPos(f, ys) =
=       let fun first(_, nil)     = NONE
```

Option Types

```
- fun firstPos(f, ys) =
=       let fun first(_, nil)     = NONE
=
```

Option Types

```
- fun firstPos(f, ys) =
=       let fun first(_, nil)     = NONE
=             | first(i, y :: ys) =
```

Option Types

```
- fun firstPos(f, ys) =
=       let fun first(_, nil)     = NONE
=             | first(i, y :: ys) =
=
```

Option Types

```
- fun firstPos(f, ys) =
=       let fun first(_, nil)     = NONE
=             | first(i, y :: ys) =
=                 if f y
```

Option Types

```
- fun firstPos(f, ys) =
=       let fun first(_, nil)     = NONE
=             | first(i, y :: ys) =
=                 if f y
=
```

Option Types

```
- fun firstPos(f, ys) =
=       let fun first(_, nil)     = NONE
=             | first(i, y :: ys) =
=                 if f y
=                 then SOME i
```

Option Types

```
- fun firstPos(f, ys) =
=       let fun first(_, nil)     = NONE
=             | first(i, y :: ys) =
=                 if f y
=                 then SOME i
=
```

Option Types

```
- fun firstPos(f, ys) =
=       let fun first(_, nil)     = NONE
=             | first(i, y :: ys) =
=                 if f y
=                 then SOME i
=                 else first(i + 1, ys)
```

Option Types

```
- fun firstPos(f, ys) =
=       let fun first(_, nil)     = NONE
=             | first(i, y :: ys) =
=                 if f y
=                 then SOME i
=                 else first(i + 1, ys)
=
```

Option Types

```
- fun firstPos(f, ys) =
=       let fun first(_, nil)     = NONE
=             | first(i, y :: ys) =
=                 if f y
=                 then SOME i
=                 else first(i + 1, ys)
=       in first(0, ys) end;
```

Option Types

```
- fun firstPos(f, ys) =
=       let fun first(_, nil)     = NONE
=             | first(i, y :: ys) =
=                 if f y
=                 then SOME i
=                 else first(i + 1, ys)
=       in first(0, ys) end;
val firstPos = fn :
  ('a -> bool) * 'a list -> int option
-
```

Option Types

```
- fun firstPos(f, ys) =
=       let fun first(_, nil)     = NONE
=             | first(i, y :: ys) =
=                 if f y
=                 then SOME i
=                 else first(i + 1, ys)
=       in first(0, ys) end;
val firstPos = fn :
  ('a -> bool) * 'a list -> int option
- firstPos(fn x => x = 4, [1, 3, 4, 5, 4, 7]);
```

Option Types

```
- fun firstPos(f, ys) =
=       let fun first(_, nil)     = NONE
=             | first(i, y :: ys) =
=                 if f y
=                 then SOME i
=                 else first(i + 1, ys)
=       in first(0, ys) end;
val firstPos = fn :
  ('a -> bool) * 'a list -> int option
- firstPos(fn x => x = 4, [1, 3, 4, 5, 4, 7]);
val it = SOME 2 : int option
-
```

Option Types

```
- fun firstPos(f, ys) =
=       let fun first(_, nil)     = NONE
=             | first(i, y :: ys) =
=                 if f y
=                 then SOME i
=                 else first(i + 1, ys)
=       in first(0, ys) end;
val firstPos = fn :
  ('a -> bool) * 'a list -> int option
- firstPos(fn x => x = 4, [1, 3, 4, 5, 4, 7]);
val it = SOME 2 : int option
- firstPos(fn x => x > 7, [1, 3, 2, 7]);
```

Option Types

```
- fun firstPos(f, ys) =
=       let fun first(_, nil)     = NONE
=             | first(i, y :: ys) =
=                 if f y
=                 then SOME i
=                 else first(i + 1, ys)
=       in first(0, ys) end;
val firstPos = fn :
  ('a -> bool) * 'a list -> int option
- firstPos(fn x => x = 4, [1, 3, 4, 5, 4, 7]);
val it = SOME 2 : int option
- firstPos(fn x => x > 7, [1, 3, 2, 7]);
val it = NONE : int option
```

Datatypes

-

Datatypes

```
- datatype tree =
```

Datatypes

```
- datatype tree =
=
```

Datatypes

```
- datatype tree =
=            Leaf of int
```

Datatypes

```
- datatype tree =
=            Leaf of int
=
```

Datatypes

```
- datatype tree =
=            Leaf of int
=          | Node of bool * tree * tree;
```

Datatypes

```
- datatype tree =
=            Leaf of int
=          | Node of bool * tree * tree;
datatype tree
  = Leaf of int | Node of bool * tree * tree
-
```

Datatypes

```
- datatype tree =
=            Leaf of int
=          | Node of bool * tree * tree;
datatype tree
  = Leaf of int | Node of bool * tree * tree
- val tr =
```

Datatypes

```
- datatype tree =
=            Leaf of int
=          | Node of bool * tree * tree;
datatype tree
  = Leaf of int | Node of bool * tree * tree
- val tr =
=
```

Datatypes

```
- datatype tree =
=            Leaf of int
=          | Node of bool * tree * tree;
datatype tree
  = Leaf of int | Node of bool * tree * tree
- val tr =
=       Node(false,
```

Datatypes

```
- datatype tree =
=            Leaf of int
=          | Node of bool * tree * tree;
datatype tree
  = Leaf of int | Node of bool * tree * tree
- val tr =
=       Node(false,
=
```

Datatypes

```
- datatype tree =
=            Leaf of int
=          | Node of bool * tree * tree;
datatype tree
  = Leaf of int | Node of bool * tree * tree
- val tr =
=       Node(false,
=            Node(true, Leaf 0, Leaf 1),
```

Datatypes

```
- datatype tree =
=            Leaf of int
=          | Node of bool * tree * tree;
datatype tree
  = Leaf of int | Node of bool * tree * tree
- val tr =
=       Node(false,
=            Node(true, Leaf 0, Leaf 1),
=
```

Datatypes

```
- datatype tree =
=            Leaf of int
=          | Node of bool * tree * tree;
datatype tree
  = Leaf of int | Node of bool * tree * tree
- val tr =
=       Node(false,
=            Node(true, Leaf 0, Leaf 1),
=            Leaf 2);
```

Datatypes

```
- datatype tree =
=            Leaf of int
=          | Node of bool * tree * tree;
datatype tree
  = Leaf of int | Node of bool * tree * tree
- val tr =
=       Node(false,
=            Node(true, Leaf 0, Leaf 1),
=            Leaf 2);
val tr =
  Node (false,Node (true,Leaf 0,Leaf 1),Leaf 2) :
  tree
-
```

Datatypes

```
- datatype tree =
=            Leaf of int
=          | Node of bool * tree * tree;
datatype tree
  = Leaf of int | Node of bool * tree * tree
- val tr =
=       Node(false,
=            Node(true, Leaf 0, Leaf 1),
=            Leaf 2);
val tr =
  Node (false,Node (true,Leaf 0,Leaf 1),Leaf 2) :
  tree
- fun size(Leaf _)            = 1
```

Datatypes

```
- datatype tree =
=            Leaf of int
=          | Node of bool * tree * tree;
datatype tree
  = Leaf of int | Node of bool * tree * tree
- val tr =
=       Node(false,
=            Node(true, Leaf 0, Leaf 1),
=            Leaf 2);
val tr =
  Node (false,Node (true,Leaf 0,Leaf 1),Leaf 2) :
  tree
- fun size(Leaf _)            = 1
=
```

Datatypes

```
- datatype tree =
=            Leaf of int
=          | Node of bool * tree * tree;
datatype tree
  = Leaf of int | Node of bool * tree * tree
- val tr =
=       Node(false,
=            Node(true, Leaf 0, Leaf 1),
=            Leaf 2);
val tr =
  Node (false,Node (true,Leaf 0,Leaf 1),Leaf 2) :
  tree
- fun size(Leaf _)             = 1
=   | size(Node(_, tr1, tr2)) =
```

Datatypes

```
- datatype tree =
=            Leaf of int
=          | Node of bool * tree * tree;
datatype tree
  = Leaf of int | Node of bool * tree * tree
- val tr =
=       Node(false,
=            Node(true, Leaf 0, Leaf 1),
=            Leaf 2);
val tr =
  Node (false,Node (true,Leaf 0,Leaf 1),Leaf 2) :
  tree
- fun size(Leaf _)             = 1
=   | size(Node(_, tr1, tr2)) =
=
```

Datatypes

```
- datatype tree =
=            Leaf of int
=          | Node of bool * tree * tree;
datatype tree
  = Leaf of int | Node of bool * tree * tree
- val tr =
=       Node(false,
=            Node(true, Leaf 0, Leaf 1),
=            Leaf 2);
val tr =
  Node (false,Node (true,Leaf 0,Leaf 1),Leaf 2) :
  tree
- fun size(Leaf _)             = 1
=   | size(Node(_, tr1, tr2)) =
=       1 + size tr1 + size tr2;
```

Datatypes

```
- datatype tree =
=            Leaf of int
=          | Node of bool * tree * tree;
datatype tree
  = Leaf of int | Node of bool * tree * tree
- val tr =
=       Node(false,
=            Node(true, Leaf 0, Leaf 1),
=            Leaf 2);
val tr =
  Node (false,Node (true,Leaf 0,Leaf 1),Leaf 2) :
  tree
- fun size(Leaf _)             = 1
=   | size(Node(_, tr1, tr2)) =
=       1 + size tr1 + size tr2;
val size = fn : tree -> int
-
```

Datatypes

```
- datatype tree =
=            Leaf of int
=          | Node of bool * tree * tree;
datatype tree
  = Leaf of int | Node of bool * tree * tree
- val tr =
=       Node(false,
=            Node(true, Leaf 0, Leaf 1),
=            Leaf 2);
val tr =
  Node (false,Node (true,Leaf 0,Leaf 1),Leaf 2) :
  tree
- fun size(Leaf _)             = 1
=   | size(Node(_, tr1, tr2)) =
=       1 + size tr1 + size tr2;
val size = fn : tree -> int
- size tr;
```

Datatypes

```
- datatype tree =
=            Leaf of int
=          | Node of bool * tree * tree;
datatype tree
  = Leaf of int | Node of bool * tree * tree
- val tr =
=       Node(false,
=            Node(true, Leaf 0, Leaf 1),
=            Leaf 2);
val tr =
  Node (false,Node (true,Leaf 0,Leaf 1),Leaf 2) :
  tree
- fun size(Leaf _)             = 1
=   | size(Node(_, tr1, tr2)) =
=       1 + size tr1 + size tr2;
val size = fn : tree -> int
- size tr;
val it = 5 : int
```

Extended Example: Generating Primes

Let's consider the problem of generating the first *n* prime numbers.

Extended Example: Generating Primes

Let's consider the problem of generating the first n prime numbers.

The key to generating primes semi-efficiently is this fact:

> *Suppose $n \in \mathbb{N}$ is at least 2. Then n is prime iff there is no $m \in \mathbb{N}$ such that*
>
> - $m < n$,
> - *n is divisible by m, and*

Extended Example: Generating Primes

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> - *m is prime.*

Extended Example: Generating Primes

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The key to generating primes semi-efficiently is this fact:

> *Suppose $n \in \mathbb{N}$ is at least 2. Then n is prime iff there is no $m \in \mathbb{N}$ such that*
>
> - $m < n$,
> - *n is divisible by m, and*
> - *m is prime.*

This holds because every natural number $n \geq 2$ can be expressed (uniquely) as a product of prime numbers (assuming n is not prime, these prime numbers will be $<$ n).

Extended Example: Generating Primes

So to test whether $n \geq 2$ is prime, we can work through the prime numbers smaller than n, from to , *rejecting* n as soon as we find a divisor of n.

Extended Example: Generating Primes

So to test whether $n \geq 2$ is prime, we can work through the prime numbers smaller than n, from *smallest* to , *rejecting* n as soon as we find a divisor of n.

Extended Example: Generating Primes

So to test whether $n \geq 2$ is prime, we can work through the prime numbers smaller than n, from *smallest* to *largest*, *rejecting* n as soon as we find a divisor of n.

Extended Example: Generating Primes

So to test whether $n \geq 2$ is prime, we can work through the prime numbers smaller than n, from *smallest* to *largest*, *rejecting* n as soon as we find a divisor of n.

Furthermore, as soon as we get to a prime m such that , we can stop and *accept* n, because if n had a prime divisor p such that $m \leq p < n$, then it would also have a prime divisor less-than m, and so would already have been rejected.

Extended Example: Generating Primes

So to test whether $n \geq 2$ is prime, we can work through the prime numbers smaller than n, from *smallest* to *largest*, *rejecting* n as soon as we find a divisor of n.

Furthermore, as soon as we get to a prime m such that $m * m > n$, we can stop and *accept* n, because if n had a prime divisor p such that $m \leq p < n$, then it would also have a prime divisor less-than m, and so would already have been rejected.

Primes in C

```
void gen_primes(int n, int *primes) {
  int i, j;
  int next = 2;  /* next candidate */
  for (i = 0; i < n; i++) {
    int found = 0;
    while (!found) {
      for (j = 0; j < i; j++) {
        int p = primes[j];
        if (next % p == 0) { break; }
        else if (p * p > next) { j = i; break; }
      }
      if (j == i) { found = 1; }
      else { next++; }
    }
    primes[i] = next++;
  }
}
```

Primes in SML: I

```
fun next(ms, l) =
      if List.exists (fn m => l mod m = 0) ms
      then next(ms, l + 1)
      else l

fun prs 0 = nil
  | prs 1 = [2]
  | prs n =
      let val ms = prs(n - 1)
      in next(ms, hd ms + 1) :: ms end

fun primes n = rev(prs n)
```

Primes in SML: II

```
fun next(ms, l) =
      if List.exists (fn m => l mod m = 0) ms
      then next(ms, l + 1)
      else l

fun prs(n, i, ms) =
      if i = n
      then rev ms
      else prs(n, i + 1, next(ms, hd ms + 1) :: ms)

fun primes n = if n = 0 then nil else prs(n, 1, [2])
```

Primes in SML: III

```
fun divisible(_, nil)     = false
  | divisible(l, m :: ms) =
      l mod m = 0 orelse
      (m * m < l andalso divisible(l, ms))

fun next(ms, l) =
      if divisible(l, ms)
      then next(ms, l + 1)
      else l

fun prs(n, i, ms, m) =
      if i = n
      then ms
      else let val k = next(ms, m + 1)
           in prs(n, i + 1, ms @ [k], k) end

fun primes n = if n = 0 then nil else prs(n, 1, [2], 2)
```

Primes in SML: IV

```
fun divisible(_, nil)     = false
  | divisible(l, m :: ms) =
      l mod m = 0 orelse
      (m * m < l andalso divisible(l, ms))

fun next(ms, p, ls, k) =
      let val (ms, p, ls) =
                if null ls orelse p >= k
                then (ms, p, ls)
                else (ms @ rev ls, hd ls * hd ls, nil)
      in if divisible(k, ms)
         then next(ms, p, ls, k + 1)
         else (ms, p, ls, k)
      end
```

Primes in SML: IV

```
fun divisible(_, nil)     = false
  | divisible(l, m :: ms) =
      l mod m = 0 orelse
      (m * m < l andalso divisible(l, ms))

fun next(ms, p, ls, k) =
      let val (ms, p, ls) =
                if null ls orelse p >= k
                then (ms, p, ls)
                else (ms @ rev ls, hd ls * hd ls, nil)
      in if divisible(k, ms)
         then next(ms, p, ls, k + 1)
         else (ms, p, ls, k)
      end
```

The reorganization of `ms`/`ls` only happens rarely; e.g, when generating the first 5,000,000 primes, the reorganization only happens

Primes in SML: IV

```
fun divisible(_, nil)     = false
  | divisible(l, m :: ms) =
      l mod m = 0 orelse
      (m * m < l andalso divisible(l, ms))

fun next(ms, p, ls, k) =
      let val (ms, p, ls) =
                if null ls orelse p >= k
                then (ms, p, ls)
                else (ms @ rev ls, hd ls * hd ls, nil)
      in if divisible(k, ms)
         then next(ms, p, ls, k + 1)
         else (ms, p, ls, k)
      end
```

The reorganization of `ms`/`ls` only happens rarely; e.g, when generating the first 5,000,000 primes, the reorganization only happens five times (when `k` is 5, 10, 50, 2210 or 4870850).

Primes in SML: IV

```
fun prs(n, i, ms, p, ls, k) =
      if i = n
      then ms @ rev ls
      else let val (ms, p, ls, k) = next(ms, p, ls, k + 1)
           in prs(n, i + 1, ms, p, k :: ls, k) end

fun primes n =
      if n = 0 then nil else prs(n, 1, [2], 4, [], 2)
```

Primes in SML: V

```
signature PRIMES =
sig

val primes : int -> int list

end;
```

Primes in SML: V

```
structure Primes :> PRIMES =
struct

fun divisible(_, nil)     = false
  | divisible(l, m :: ms) =
      l mod m = 0 orelse
      (m * m < l andalso divisible(l, ms))

fun next(ms, p, ls, k) =
      let val (ms, p, ls) =
                if null ls orelse p >= k
                then (ms, p, ls)
                else (ms @ rev ls, hd ls * hd ls, nil)
      in if divisible(k, ms)
         then next(ms, p, ls, k + 1)
         else (ms, p, ls, k)
      end
```

Primes in SML: V

```
fun prs(n, i, ms, p, ls, k) =
      if i = n
      then ms @ rev ls
      else let val (ms, p, ls, k) = next(ms, p, ls, k + 1)
           in prs(n, i + 1, ms, p, k :: ls, k) end

fun primes n =
      if n = 0 then nil else prs(n, 1, [2], 4, [], 2)

end;
```

Comparison Generating First 5,000,000 Primes

```
$ time primes-gcc 5000000 > /tmp/primes-gcc

real 0m16.989s
user 0m16.804s
sys 0m0.058s
$ time primes-smlnj 5000000 > /tmp/primes-smlnj

real 1m12.238s
user 1m10.816s
sys 0m1.360s
$ time primes-mlton 5000000 > /tmp/primes-mlton

real 0m49.371s
user 0m48.896s
sys 0m0.376s
```

Comparison Generating First 5,000,000 Primes

```
$ cmp /tmp/primes-gcc /tmp/primes-smlnj
$ cmp /tmp/primes-smlnj /tmp/primes-mlton
$ wc -l /tmp/primes-mlton
 5000000 /tmp/primes-mlton
$ tail /tmp/primes-mlton
86027987
86027999
86028011
86028037
86028049
86028053
86028097
86028101
86028113
86028121
```

Primes in SML: VI

```
signature PRIMES =
sig

type state

val init : state

val next : state -> int * state

end;
```

Primes in SML: VI

```
structure Primes :> PRIMES =
struct

type state = int list * int * int list * int

(* invariant on state of the form (ms, p, ls, k):
   ms @ rev ls is the first length ms + length ns
   primes in ascending order, ms is nonempty,
   p is the square of the last element of ms, and
   k is the last element of ms @ rev ls *)

val init = ([2], 4, [], 2)

fun divisible(_, nil)     = false
  | divisible(l, m :: ms) =
      l mod m = 0 orelse
      (m * m < l andalso divisible(l, ms))
```

Primes in SML: VI

```
fun nxt(ms, p, ls, k) =
      let val (ms, p, ls) =
                if null ls orelse p >= k
                then (ms, p, ls)
                else (ms @ rev ls, hd ls * hd ls, nil)
      in if divisible(k, ms)
         then nxt(ms, p, ls, k + 1)
         else (ms, p, ls, k)
      end

fun next ((ms, p, ls, k) : state) : int * state =
      (k,
       let val (ms, p, ls, k) = nxt(ms, p, ls, k + 1)
           in (ms, p, k :: ls, k) end)

end;
```

Primes in SML: VI

-

Primes in SML: VI

```
- val (n, st) = Primes.next Primes.init;
```

Primes in SML: VI

```
- val (n, st) = Primes.next Primes.init;
val n = 2 : int
val st = - : Primes.state
-
```

Primes in SML: VI

```
- val (n, st) = Primes.next Primes.init;
val n = 2 : int
val st = - : Primes.state
- val (n, st) = Primes.next st;
```

Primes in SML: VI

```
- val (n, st) = Primes.next Primes.init;
val n = 2 : int
val st = - : Primes.state
- val (n, st) = Primes.next st;
val n = 3 : int
val st = - : Primes.state
-
```

Primes in SML: VI

```
- val (n, st) = Primes.next Primes.init;
val n = 2 : int
val st = - : Primes.state
- val (n, st) = Primes.next st;
val n = 3 : int
val st = - : Primes.state
- val (n, st) = Primes.next st;
```

Primes in SML: VI

```
- val (n, st) = Primes.next Primes.init;
val n = 2 : int
val st = - : Primes.state
- val (n, st) = Primes.next st;
val n = 3 : int
val st = - : Primes.state
- val (n, st) = Primes.next st;
val n = 5 : int
val st = - : Primes.state
-
```

Primes in SML: VI

```
- val (n, st) = Primes.next Primes.init;
val n = 2 : int
val st = - : Primes.state
- val (n, st) = Primes.next st;
val n = 3 : int
val st = - : Primes.state
- val (n, st) = Primes.next st;
val n = 5 : int
val st = - : Primes.state
- val (n, st) = Primes.next st;
```

Primes in SML: VI

```
- val (n, st) = Primes.next Primes.init;
val n = 2 : int
val st = - : Primes.state
- val (n, st) = Primes.next st;
val n = 3 : int
val st = - : Primes.state
- val (n, st) = Primes.next st;
val n = 5 : int
val st = - : Primes.state
- val (n, st) = Primes.next st;
val n = 7 : int
val st = - : Primes.state
-
```

Primes in SML: VI

```
- val (n, st) = Primes.next Primes.init;
val n = 2 : int
val st = - : Primes.state
- val (n, st) = Primes.next st;
val n = 3 : int
val st = - : Primes.state
- val (n, st) = Primes.next st;
val n = 5 : int
val st = - : Primes.state
- val (n, st) = Primes.next st;
val n = 7 : int
val st = - : Primes.state
- val (n, st) = Primes.next st;
```

Primes in SML: VI

```
- val (n, st) = Primes.next Primes.init;
val n = 2 : int
val st = - : Primes.state
- val (n, st) = Primes.next st;
val n = 3 : int
val st = - : Primes.state
- val (n, st) = Primes.next st;
val n = 5 : int
val st = - : Primes.state
- val (n, st) = Primes.next st;
val n = 7 : int
val st = - : Primes.state
- val (n, st) = Primes.next st;
val n = 11 : int
val st = - : Primes.state
```

Streams

Can we create an infinite list (stream) of all (until we have integer overflow) primes like this?

```
fun g state =
      let (x, state) = Primes.next state
      in x :: g state end;
val primes = g Primes.init;
```

Streams

Can we create an infinite list (stream) of all (until we have integer overflow) primes like this?

```
fun g state =
      let (x, state) = Primes.next state
      in x :: g state end;
val primes = g Primes.init;
```

No, only with *lazy evaluation*, where infinite streams can be created, and the part of a stream that is visited is *memoized*.

Streams

Can we create an infinite list (stream) of all (until we have integer overflow) primes like this?

```
fun g state =
      let (x, state) = Primes.next state
      in x :: g state end;
val primes = g Primes.init;
```

No, only with *lazy evaluation*, where infinite streams can be created, and the part of a stream that is visited is *memoized*.

We can simulate lazy evaluation in SML using *thunks* and *references*.

Suspensions

```
signature SUSP =
sig

type 'a susp

val delay : (unit -> 'a) -> 'a susp

val force : 'a susp -> 'a

end
```

The only value of type `unit` is `()`.

References

```
type 'a ref

val ref : 'a -> 'a ref
val !   : 'a ref -> 'a
val :=  : 'a ref * 'a -> unit
```

References

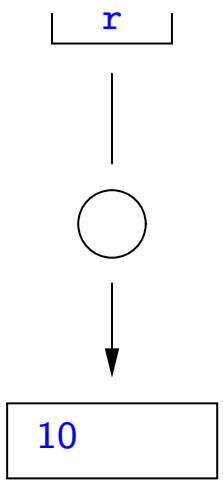
–

References

```
- val r = ref 10;
```

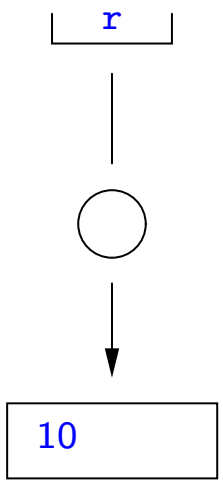
References

```
- val r = ref 10;
val r = ref 10 : int ref
-
```



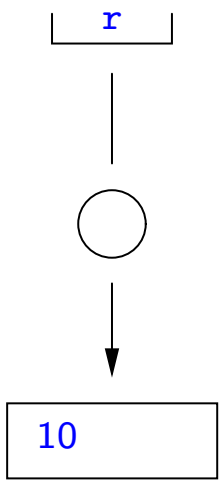
References

```
- val r = ref 10;
val r = ref 10 : int ref
- !r;
```



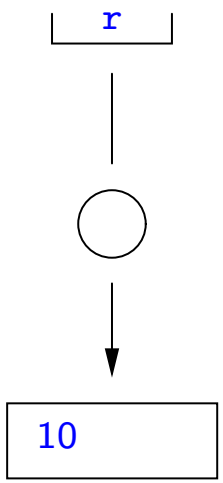
References

```
- val r = ref 10;
val r = ref 10 : int ref
- !r;
val it = 10 : int
-
```



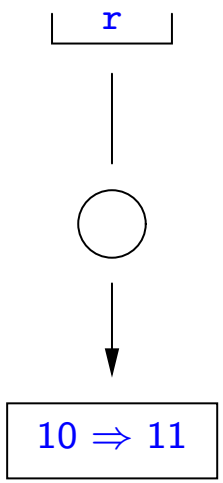
References

```
- val r = ref 10;
val r = ref 10 : int ref
- !r;
val it = 10 : int
- r := !r + 1;
```



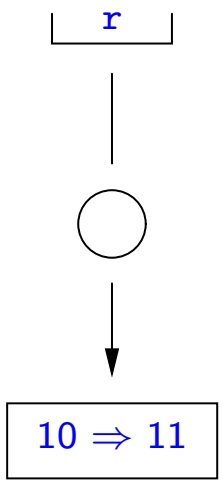
References

```
- val r = ref 10;
val r = ref 10 : int ref
- !r;
val it = 10 : int
- r := !r + 1;
val it = () : unit
-
```



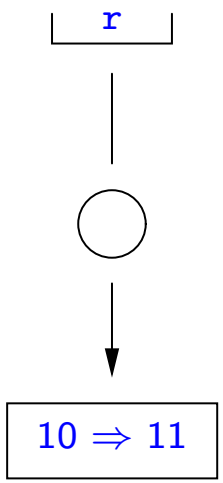
References

```
- val r = ref 10;
val r = ref 10 : int ref
- !r;
val it = 10 : int
- r := !r + 1;
val it = () : unit
- !r;
```



References

```
- val r = ref 10;
val r = ref 10 : int ref
- !r;
val it = 10 : int
- r := !r + 1;
val it = () : unit
- !r;
val it = 11 : int
```



Suspensions

```
structure Susp :> SUSP =
struct

datatype 'a delay = Value of 'a
                  | Delay of unit -> 'a

type 'a susp = 'a delay ref

fun delay f = ref(Delay f)

fun force(ref(Value x))      = x
  | force(r as ref(Delay f)) =
      let val x = f()
      in r := Value x; x end

end;
```

Streams

```
signature STREAM =
sig

type 'a stream

val make : 'a * ('a -> 'b * 'a) -> 'b stream

val get : 'a stream -> 'a * 'a stream

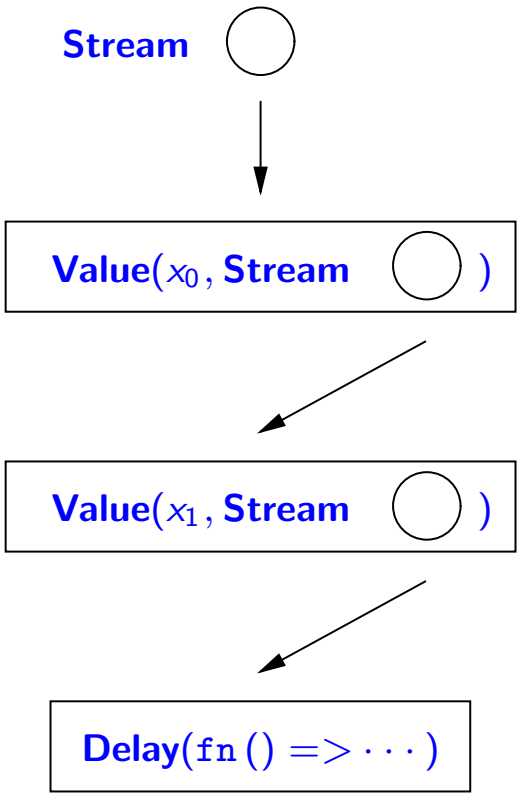
val takeToList : 'a stream * int -> 'a list

val drop : 'a stream * int -> 'a stream

val rangeToList : 'a stream * int * int -> 'a list

end;
```

Streams



Streams

```
structure Stream :> STREAM =
struct

datatype 'a stream = Stream of ('a * 'a stream)Susp.susp

fun make(state, f) =
      let fun g state =
                Stream
                (Susp.delay
                 (fn () =>
                       let val (x, state) = f state
                       in (x, g state) end))
      in g state end

fun get(Stream x) = Susp.force x
```

Streams

```
fun takeToList(stm, n) =
      let fun tke(stm, n, xs) =
                  if n <= 0
                  then rev xs
                  else let val (x, stm) = get stm
                       in tke(stm, n - 1, x :: xs) end
      in tke(stm, n, nil) end

fun drop(stm, n) =
      if n <= 0
      then stm
      else let val (_, stm) = get stm
           in drop(stm, n - 1) end
```

Streams

```
fun rangeToList(stm, n, m) =
      if n <= 0 orelse m < n
      then nil
      else takeToList(drop(stm, n - 1), m - n + 1)

end;
```

Primes in SML: VI

-

Primes in SML: VI

```
- val primes = Stream.make(Primes.init, Primes.next);
```

Primes in SML: VI

```
- val primes = Stream.make(Primes.init, Primes.next);
val primes = - : int Stream.stream
-
```

Primes in SML: VI

```
- val primes = Stream.make(Primes.init, Primes.next);
val primes = - : int Stream.stream
- val stm = primes;
```

Primes in SML: VI

```
- val primes = Stream.make(Primes.init, Primes.next);
val primes = - : int Stream.stream
- val stm = primes;
val stm = - : int Stream.stream
-
```

Primes in SML: VI

```
- val primes = Stream.make(Primes.init, Primes.next);
val primes = - : int Stream.stream
- val stm = primes;
val stm = - : int Stream.stream
- val (x, stm) = Stream.get stm;
```

Primes in SML: VI

```
- val primes = Stream.make(Primes.init, Primes.next);
val primes = - : int Stream.stream
- val stm = primes;
val stm = - : int Stream.stream
- val (x, stm) = Stream.get stm;
val x = 2 : int
val stm = - : int Stream.stream
-
```

Primes in SML: VI

```
- val primes = Stream.make(Primes.init, Primes.next);
val primes = - : int Stream.stream
- val stm = primes;
val stm = - : int Stream.stream
- val (x, stm) = Stream.get stm;
val x = 2 : int
val stm = - : int Stream.stream
- val (x, stm) = Stream.get stm;
```

Primes in SML: VI

```
- val primes = Stream.make(Primes.init, Primes.next);
val primes = - : int Stream.stream
- val stm = primes;
val stm = - : int Stream.stream
- val (x, stm) = Stream.get stm;
val x = 2 : int
val stm = - : int Stream.stream
- val (x, stm) = Stream.get stm;
val x = 3 : int
val stm = - : int Stream.stream
-
```

Primes in SML: VI

```
- val primes = Stream.make(Primes.init, Primes.next);
val primes = - : int Stream.stream
- val stm = primes;
val stm = - : int Stream.stream
- val (x, stm) = Stream.get stm;
val x = 2 : int
val stm = - : int Stream.stream
- val (x, stm) = Stream.get stm;
val x = 3 : int
val stm = - : int Stream.stream
- val (x, stm) = Stream.get stm;
```

Primes in SML: VI

```
- val primes = Stream.make(Primes.init, Primes.next);
val primes = - : int Stream.stream
- val stm = primes;
val stm = - : int Stream.stream
- val (x, stm) = Stream.get stm;
val x = 2 : int
val stm = - : int Stream.stream
- val (x, stm) = Stream.get stm;
val x = 3 : int
val stm = - : int Stream.stream
- val (x, stm) = Stream.get stm;
val x = 5 : int
val stm = - : int Stream.stream
-
```

Primes in SML: VI

```
- val primes = Stream.make(Primes.init, Primes.next);
val primes = - : int Stream.stream
- val stm = primes;
val stm = - : int Stream.stream
- val (x, stm) = Stream.get stm;
val x = 2 : int
val stm = - : int Stream.stream
- val (x, stm) = Stream.get stm;
val x = 3 : int
val stm = - : int Stream.stream
- val (x, stm) = Stream.get stm;
val x = 5 : int
val stm = - : int Stream.stream
- val (x, _) = Stream.get primes;
```

Primes in SML: VI

```
- val primes = Stream.make(Primes.init, Primes.next);
val primes = - : int Stream.stream
- val stm = primes;
val stm = - : int Stream.stream
- val (x, stm) = Stream.get stm;
val x = 2 : int
val stm = - : int Stream.stream
- val (x, stm) = Stream.get stm;
val x = 3 : int
val stm = - : int Stream.stream
- val (x, stm) = Stream.get stm;
val x = 5 : int
val stm = - : int Stream.stream
- val (x, _) = Stream.get primes;
val x = 2 : int
```

Primes in SML: VI

-

Primes in SML: VI

```
- Stream.rangeToList(primes, 1, 10);
```

Primes in SML: VI

```
- Stream.rangeToList(primes, 1, 10);
val it = [2,3,5,7,11,13,17,19,23,29] : int list
-
```

Primes in SML: VI

```
- Stream.rangeToList(primes, 1, 10);
val it = [2,3,5,7,11,13,17,19,23,29] : int list
- Stream.rangeToList(primes, 50000, 50010);
```

Primes in SML: VI

```
- Stream.rangeToList(primes, 1, 10);
val it = [2,3,5,7,11,13,17,19,23,29] : int list
- Stream.rangeToList(primes, 50000, 50010);
val it =
  [611953,611957,611969,611977,611993,611999,612011,
   612023,612037,612041,612043] : int list
-
```

Primes in SML: VI

```
- Stream.rangeToList(primes, 1, 10);
val it = [2,3,5,7,11,13,17,19,23,29] : int list
- Stream.rangeToList(primes, 50000, 50010);
val it =
  [611953,611957,611969,611977,611993,611999,612011,
   612023,612037,612041,612043] : int list
- Stream.rangeToList(primes, 40000, 40010);
```

Primes in SML: VI

```
- Stream.rangeToList(primes, 1, 10);
val it = [2,3,5,7,11,13,17,19,23,29] : int list
- Stream.rangeToList(primes, 50000, 50010);
val it =
  [611953,611957,611969,611977,611993,611999,612011,
   612023,612037,612041,612043] : int list
- Stream.rangeToList(primes, 40000, 40010);
val it =
  [479909,479939,479951,479953,479957,479971,480013,
   480017,480019,480023,480043] : int list
```

Primes in SML: VI

Twin primes are primes that are separated by 2, like 3/5, 5/7 and 11/13. We can write a *stream transformer* that turns the stream of all primes into the stream of all twin pairs:

Primes in SML: VI

Twin primes are primes that are separated by 2, like 3/5, 5/7 and 11/13. We can write a *stream transformer* that turns the stream of all primes into the stream of all twin pairs:

```
signature TWINS =
sig

val twins : int Stream.stream -> (int * int) Stream.stream

end;
```

Primes in SML: VI

```
structure Twins :> TWINS =
struct

fun twins stm =
      let val init : int * int Stream.stream =
                Stream.get stm

          fun next (n, stm) =
                let val (m, stm) = Stream.get stm
                in if m = n + 2
                   then ((n, m), (m, stm))
                   else next (m, stm)
                end
      in Stream.make(init, next) end

end;
```

Primes in SML: VI

```
-
```

Primes in SML: VI

```
- val twins = Twins.twins primes;
```

Primes in SML: VI

```
- val twins = Twins.twins primes;
val twins = - : (int * int) Stream.stream
-
```

Primes in SML: VI

```
- val twins = Twins.twins primes;
val twins = - : (int * int) Stream.stream
- Stream.rangeToList(twins, 1, 10);
```

Primes in SML: VI

```
- val twins = Twins.twins primes;
val twins = - : (int * int) Stream.stream
- Stream.rangeToList(twins, 1, 10);
val it =
  [(3,5),(5,7),(11,13),(17,19),(29,31),(41,43),
   (59,61),(71,73),(101,103),(107,109)] :
  (int * int) list
-
```

Primes in SML: VI

```
- val twins = Twins.twins primes;
val twins = - : (int * int) Stream.stream
- Stream.rangeToList(twins, 1, 10);
val it =
  [(3,5),(5,7),(11,13),(17,19),(29,31),(41,43),
   (59,61),(71,73),(101,103),(107,109)] :
  (int * int) list
- Stream.rangeToList(twins, 10000, 10010);
```

Primes in SML: VI

```
- val twins = Twins.twins primes;
val twins = - : (int * int) Stream.stream
- Stream.rangeToList(twins, 1, 10);
val it =
  [(3,5),(5,7),(11,13),(17,19),(29,31),(41,43),
   (59,61),(71,73),(101,103),(107,109)] :
  (int * int) list
- Stream.rangeToList(twins, 10000, 10010);
val it =
  [(1260989,1260991),(1261079,1261081),
   (1261259,1261261),(1261487,1261489),
   (1261697,1261699),(1261829,1261831),
   (1261889,1261891),(1262081,1262083),
   (1262099,1262101),(1262291,1262293),
   (1262621,1262623)] : (int * int) list
```