CIS 570 — Introduction to Formal Language Theory — Fall 2006

Exercise Set 1

Model Answers

Exercise 1

We proceed by mathematical induction.

(Basis Step) We have that $3(0^2 + 0 + 2) = 3 * 2 = 6 = 6 * 1$ and $1 \in \mathbb{N}$.

(Inductive Step) Suppose $n \in \mathbb{N}$, and assume the inductive hypothesis:

$$3(n^2 + n + 2) = 6m, \text{ for some } m \in \mathbb{N}.$$

We must show that,

$$3((n+1)^2 + (n+1) + 2) = 6m, \text{ for some } m \in \mathbb{N}.$$

We have that

$$\begin{aligned} 3((n+1)^2 + (n+1) + 2) &= 3(n^2 + 2n + 1) + 3(n+1) + 6 \\ &= 3n^2 + 6n + 3 + 3n + 3 + 6 \\ &= 3n^2 + 3n + 6 + 6n + 6 \\ &= 3(n^2 + n + 2) + 6(n+1) \\ &= 6m + 6(n+1) && \text{(inductive hypothesis)} \\ &= 6(m + n + 1). \end{aligned}$$

Thus $3((n+1)^2 + (n+1) + 2) = 6(m + n + 1)$ and $m + n + 1 \in \mathbb{N}$.

Exercise 2

We proceed by strong induction. Suppose $n \in \mathbb{N}$, and assume the inductive hypothesis: for all $m \in \mathbb{N}$, if $m < n$, then,

$$\text{if } m \geq 18, \text{ then there are } i, j \in \mathbb{N} \text{ such that } m = 4i + 7j.$$

We must show that,

$$\text{if } n \geq 18, \text{ then there are } i, j \in \mathbb{N} \text{ such that } n = 4i + 7j.$$

Suppose $n \geq 18$. We must show that there are $i, j \in \mathbb{N}$ such that $n = 4i + 7j$. There are five cases to consider.

- Suppose $n = 18$. Then $n = 18 = 4 * 1 + 7 * 2$ and $1, 2 \in \mathbb{N}$.
- Suppose $n = 19$. Then $n = 19 = 4 * 3 + 7 * 1$ and $3, 1 \in \mathbb{N}$.

- Suppose $n = 20$. Then $n = 20 = 4 * 5 + 7 * 0$ and $5, 0 \in \mathbb{N}$.
- Suppose $n = 21$. Then $n = 21 = 4 * 0 + 7 * 3$ and $0, 3 \in \mathbb{N}$.
- Suppose $n \geq 22$. Thus $18 \leq n - 4 < n$. Because $n - 4 < n$, the inductive hypothesis tells us that

  $$\text{if } n - 4 \geq 18, \text{ then there are } i, j \in \mathbb{N} \text{ such that } n - 4 = 4i + 7j.$$

  But $n - 4 \geq 18$, and thus $n - 4 = 4i + 7j$ for some $i, j \in \mathbb{N}$. Hence

  $$n = (n - 4) + 4 = 4i + 7j + 4 = 4(i + 1) + 7j,$$

  and $i + 1, j \in \mathbb{N}$.

Exercise 3

(a) Suppose A, B and C are sets. We must show that

$$A - (B \cup C) = (A - B) - C.$$

It will suffice to show that

$$A - (B \cup C) \subseteq (A - B) - C \subseteq A - (B \cup C).$$

($A - (B \cup C) \subseteq (A - B) - C$) Suppose $w \in A - (B \cup C)$. We must show that $w \in (A - B) - C$. By the assumption, we have that $w \in A$ and $w \notin (B \cup C)$.

Suppose, toward a contradiction, that $w \in B$. Then $w \in B \cup C$—contradiction. Thus $w \notin B$.

Suppose, toward a contradiction, that $w \in C$. Then $w \in B \cup C$—contradiction. Thus $w \notin C$.

Because $w \in A$ and $w \notin B$, we have that $w \in A - B$. Then, since $w \notin C$, it follows that $w \in (A - B) - C$.

($(A - B) - C \subseteq A - (B \cup C)$) Suppose $w \in (A - B) - C$. We must show that $w \in A - (B \cup C)$. By the assumption, we have that $w \in A - B$ and $w \notin C$. Hence $w \in A$ and $w \notin B$.

Suppose, toward a contradiction, that $w \in B \cup C$. There are two cases to consider.

- Suppose $w \in B$. But $w \notin B$—contradiction.
- Suppose $w \in C$. But $w \notin C$—contradiction.

Since we obtained a contradiction in both cases, we have an overall contradiction. Thus $w \notin B \cup C$.

Because $w \in A$ and $w \notin B \cup C$, we have that $w \in A - (B \cup C)$.

(b) Suppose A, B and C are sets. We must show that

$$A - (B \cap C) = (A - B) \cup (A - C).$$

It will suffice to show that

$$A - (B \cap C) \subseteq (A - B) \cup (A - C) \subseteq A - (B \cap C).$$

($A - (B \cap C) \subseteq (A - B) \cup (A - C)$) Suppose $w \in A - (B \cap C)$. We must show that $w \in (A - B) \cup (A - C)$. By the assumption, we have that $w \in A$ and $w \notin B \cap C$. There are two cases to consider.

- Suppose $w \in B$. Suppose, toward a contradiction, that $w \in C$. Thus $w \in B \cap C$—contradiction. Thus $w \notin C$. And $w \in A$, and thus $w \in A - C \subseteq (A - B) \cup (A - C)$.
- Suppose $w \notin B$. Because $w \in A$, it follows that $w \in A - B \subseteq (A - B) \cup (A - C)$.

($(A - B) \cup (A - C) \subseteq A - (B \cap C)$) Suppose $w \in (A - B) \cup (A - C)$. We must show that $w \in A - (B \cap C)$. By the assumption, there are two cases to consider.

- Suppose $w \in A - B$. Hence $w \in A$ and $w \notin B$. Suppose, toward a contradiction, that $w \in B \cap C$. Thus $w \in B$—contradiction. Hence $w \notin B \cap C$, so that $w \in A - (B \cap C)$.
- Suppose $w \in A - C$. Hence $w \in A$ and $w \notin C$. Suppose, toward a contradiction, that $w \in B \cap C$. Thus $w \in C$—contradiction. Hence $w \notin B \cap C$, so that $w \in A - (B \cap C)$.

Exercise 4

(a) We use induction on X to show that, for all $w \in X$, $w \in Y$. There are four steps to show.

(1) Since $\mathbf{diff}(\%) = 0$, we have that $\% \in Y$.

(2) Suppose $w, x, y, z \in X$ and $w, x, y, z \in Y$. We must show that $w\mathsf{1}x\mathsf{1}y\mathsf{0}z \in Y$. Because $w, x, y, z \in Y$, it follows that $\mathbf{diff}(w) = \mathbf{diff}(x) = \mathbf{diff}(y) = \mathbf{diff}(z) = 0$. Hence $\mathbf{diff}(w\mathsf{1}x\mathsf{1}y\mathsf{0}z) = \mathbf{diff}(w) + \mathbf{diff}(\mathsf{1}) + \mathbf{diff}(x) + \mathbf{diff}(\mathsf{1}) + \mathbf{diff}(y) + \mathbf{diff}(\mathsf{0}) + \mathbf{diff}(z) = 0 + 1 + 0 + 1 + 0 + -2 + 0 = 0$, showing that $w\mathsf{1}x\mathsf{1}y\mathsf{0}z \in Y$.

(3) Suppose $w, x, y, z \in X$ and $w, x, y, z \in Y$. We must show that $w\mathsf{1}x\mathsf{0}y\mathsf{1}z \in Y$. Because $w, x, y, z \in Y$, it follows that $\mathbf{diff}(w) = \mathbf{diff}(x) = \mathbf{diff}(y) = \mathbf{diff}(z) = 0$. Hence $\mathbf{diff}(w\mathsf{1}x\mathsf{0}y\mathsf{1}z) = \mathbf{diff}(w) + \mathbf{diff}(\mathsf{1}) + \mathbf{diff}(x) + \mathbf{diff}(\mathsf{0}) + \mathbf{diff}(y) + \mathbf{diff}(\mathsf{1}) + \mathbf{diff}(z) = 0 + 1 + 0 + -2 + 0 + 1 + 0 = 0$, showing that $w\mathsf{1}x\mathsf{0}y\mathsf{1}z \in Y$.

(4) Suppose $w, x, y, z \in X$ and $w, x, y, z \in Y$. We must show that $w\mathsf{0}x\mathsf{1}y\mathsf{1}z \in Y$. Because $w, x, y, z \in Y$, it follows that $\mathbf{diff}(w) = \mathbf{diff}(x) = \mathbf{diff}(y) = \mathbf{diff}(z) = 0$. Hence $\mathbf{diff}(w\mathsf{0}x\mathsf{1}y\mathsf{1}z) = \mathbf{diff}(w) + \mathbf{diff}(\mathsf{0}) + \mathbf{diff}(x) + \mathbf{diff}(\mathsf{1}) + \mathbf{diff}(y) + \mathbf{diff}(\mathsf{1}) + \mathbf{diff}(z) = 0 + -2 + 0 + 1 + 0 + 1 + 0 = 0$, showing that $w\mathsf{0}x\mathsf{1}y\mathsf{1}z \in Y$.

(b) We begin by proving two lemmas.

Lemma ES1.4.1
For all $w \in \{\mathsf{0}, \mathsf{1}\}^*$, if $\mathbf{diff}(w) \geq 1$, then there are $x, y \in \{\mathsf{0}, \mathsf{1}\}^*$ such that $w = x\mathsf{1}y$, $\mathbf{diff}(x) = 0$ and $\mathbf{diff}(y) = \mathbf{diff}(w) - 1$.

Proof. Suppose $w \in \{\mathsf{0}, \mathsf{1}\}^*$ and suppose $\mathbf{diff}(w) \geq 1$. Let z be the shortest prefix of w such that $\mathbf{diff}(z) \geq 1$. (Such a z exists, because w is a prefix of itself with a positive diff.) Let $y \in \{\mathsf{0}, \mathsf{1}\}^*$ be such that $w = zy$. Because $\mathbf{diff}(z) \geq 1$, we have that $z \neq \%$. Thus $z = xb$, for some $x \in \{\mathsf{0}, \mathsf{1}\}^*$ and $b \in \{\mathsf{0}, \mathsf{1}\}$. Because x is a shorter prefix of w than z, it follows that $\mathbf{diff}(x) \leq 0$.

Suppose, toward a contradiction, that $b = \mathtt{0}$. Since $\textbf{diff}(x) + -2 = \textbf{diff}(x\mathtt{0}) = \textbf{diff}(xb) = \textbf{diff}(z) \geq 1$, we have that $\textbf{diff}(x) \geq 3$—contradiction. Thus $b = \mathtt{1}$, so that $z = xb = x\mathtt{1}$ and $w = zy = x\mathtt{1}y$.

Since $\textbf{diff}(x) + 1 = \textbf{diff}(x\mathtt{1}) = \textbf{diff}(z) \geq 1$, we have that $\textbf{diff}(x) \geq 0$. But $\textbf{diff}(x) \leq 0$, and thus $\textbf{diff}(x) = 0$. And, because $1 + \textbf{diff}(y) = 0 + 1 + \textbf{diff}(y) = \textbf{diff}(x\mathtt{1}y) = \textbf{diff}(w)$, we have that $\textbf{diff}(y) = \textbf{diff}(w) - 1$. $\square$

Lemma ES1.4.2
For all $w \in \{\mathtt{0},\mathtt{1}\}^*$, if $\textbf{diff}(w) \leq -1$, then there are $x, y \in \{\mathtt{0},\mathtt{1}\}^*$ such that $w = x\mathtt{0}y$, and either

- $\textbf{diff}(x) = 0$ *and* $\textbf{diff}(y) = \textbf{diff}(w) + 2$*; or*
- $\textbf{diff}(x) = 1$ *and* $\textbf{diff}(y) = \textbf{diff}(w) + 1$*.*

Proof. Suppose $w \in \{\mathtt{0},\mathtt{1}\}^*$ and suppose $\textbf{diff}(w) \leq -1$. Let z be the shortest prefix of w such that $\textbf{diff}(z) \leq -1$. (Such a z exists, because w is a prefix of itself with a negative diff.) Let $y \in \{\mathtt{0},\mathtt{1}\}^*$ be such that $w = zy$. Because $\textbf{diff}(z) \leq -1$, we have that $z \neq \%$. Thus $z = xb$, for some $x \in \{\mathtt{0},\mathtt{1}\}^*$ and $b \in \{\mathtt{0},\mathtt{1}\}$. Because x is a shorter prefix of w than z, it follows that $\textbf{diff}(x) \geq 0$.

Suppose, toward a contradiction, that $b = \mathtt{1}$. Since $\textbf{diff}(x)+1 = \textbf{diff}(x\mathtt{1}) = \textbf{diff}(xb) = \textbf{diff}(z) \leq -1$, we have that $\textbf{diff}(x) \leq -2$—contradiction. Thus $b = \mathtt{0}$, so that $z = xb = x\mathtt{0}$ and $w = zy = x\mathtt{0}y$.

Since $\textbf{diff}(x) + -2 = \textbf{diff}(x\mathtt{0}) = \textbf{diff}(z) \leq -1$, we have that $\textbf{diff}(x) \leq 1$. But $\textbf{diff}(x) \geq 0$, and thus $\textbf{diff}(x) \in \{0, 1\}$. Thus there are two cases to consider.

- Suppose $\textbf{diff}(x) = 0$. Because $-2 + \textbf{diff}(y) = 0 + -2 + \textbf{diff}(y) = \textbf{diff}(x\mathtt{0}y) = \textbf{diff}(w)$, we have that $\textbf{diff}(y) = \textbf{diff}(w) + 2$. Thus $\textbf{diff}(x) = 0$ and $\textbf{diff}(y) = \textbf{diff}(w) + 2$.
- Suppose $\textbf{diff}(x) = 1$. Because $-1 + \textbf{diff}(y) = 1 + -2 + \textbf{diff}(y) = \textbf{diff}(x\mathtt{0}y) = \textbf{diff}(w)$, we have that $\textbf{diff}(y) = \textbf{diff}(w) + 1$. Thus $\textbf{diff}(x) = 1$ and $\textbf{diff}(y) = \textbf{diff}(w) + 1$.

$\square$

Now, we show that $Y \subseteq X$. Since $Y \subseteq \{\mathtt{0},\mathtt{1}\}^*$, it will suffice to show that, for all $w \in \{\mathtt{0},\mathtt{1}\}^*$,

$$\text{if } w \in Y, \text{ then } w \in X.$$

We proceed by strong string induction. Suppose $w \in \{\mathtt{0},\mathtt{1}\}^*$, and assume the inductive hypothesis: for all $x \in \{\mathtt{0},\mathtt{1}\}^*$, if $|x| < |w|$, then

$$\text{if } x \in Y, \text{ then } x \in X.$$

We must show that

$$\text{if } w \in Y, \text{ then } w \in X.$$

Suppose $w \in Y$, so that $\textbf{diff}(w) = 0$. We must show that $w \in X$. There are three cases to consider.

- Suppose $w = \%$. Then $w = \% \in X$, by Part (1) of the definition of X.

- Suppose $w = \mathtt{0}s$, for some $s \in \{\mathtt{0},\mathtt{1}\}^*$. Because $-2 + \mathbf{diff}(s) = \mathbf{diff}(\mathtt{0}s) = \mathbf{diff}(w) = 0$, we have that $\mathbf{diff}(s) = 2$. Since $\mathbf{diff}(s) \geq 1$, Lemma ES1.4.1 tells us that there are $x, t \in \{\mathtt{0},\mathtt{1}\}^*$ such that $s = x\mathtt{1}t$, $\mathbf{diff}(x) = 0$ and $\mathbf{diff}(t) = \mathbf{diff}(s) - 1$. Hence $x \in Y$ and $\mathbf{diff}(t) = 2 - 1 = 1$. Since $\mathbf{diff}(t) \geq 1$, Lemma ES1.4.1 tells us that there are $y, z \in \{\mathtt{0},\mathtt{1}\}^*$ such that $t = y\mathtt{1}z$, $\mathbf{diff}(y) = 0$ and $\mathbf{diff}(z) = \mathbf{diff}(t) - 1$. Hence $y \in Y$ and $\mathbf{diff}(z) = 1 - 1 = 0$, so that $z \in Y$. Summarizing, we have that $w = \mathtt{0}s = \mathtt{0}x\mathtt{1}t = \mathtt{0}x\mathtt{1}y\mathtt{1}z$ and $x, y, z \in Y$. Because $|x| < |w|$, $|y| < |w|$ and $|z| < |w|$, the inductive hypothesis tells us that $x, y, z \in X$. By Part (1) of the definition on X, we have that $\% \in X$. Finally, since $\%, x, y, z \in X$, Part (4) of the definition of X tells us that $w = \mathtt{0}x\mathtt{1}y\mathtt{1}z = \%\mathtt{0}x\mathtt{1}y\mathtt{1}z \in X$.

- Suppose $w = \mathtt{1}s$, for some $s \in \{\mathtt{0},\mathtt{1}\}^*$. Because $1 + \mathbf{diff}(s) = \mathbf{diff}(\mathtt{1}s) = \mathbf{diff}(w) = 0$, we have that $\mathbf{diff}(s) = -1$. Since $\mathbf{diff}(s) \leq -1$, Lemma ES1.4.2 tells us that there are $x, y \in \{\mathtt{0},\mathtt{1}\}^*$ such that $s = x\mathtt{0}y$, and either

  - $\mathbf{diff}(x) = 0$ and $\mathbf{diff}(y) = \mathbf{diff}(s) + 2$; or
  - $\mathbf{diff}(x) = 1$ and $\mathbf{diff}(y) = \mathbf{diff}(s) + 1$.

  Thus there are two cases to consider.

  - Suppose $\mathbf{diff}(x) = 0$ and $\mathbf{diff}(y) = \mathbf{diff}(s) + 2$. Hence $x \in Y$ and $\mathbf{diff}(y) = -1 + 2 = 1$. Since $\mathbf{diff}(y) \geq 1$, Lemma ES1.4.1 tells us that there are $z, t \in \{\mathtt{0},\mathtt{1}\}^*$ such that $y = z\mathtt{1}t$, $\mathbf{diff}(z) = 0$ and $\mathbf{diff}(t) = \mathbf{diff}(y) - 1$. Hence $z \in Y$ and $\mathbf{diff}(t) = 1 - 1 = 0$, so that $t \in Y$. Summarizing, we have that $w = \mathtt{1}s = \mathtt{1}x\mathtt{0}y = \mathtt{1}x\mathtt{0}z\mathtt{1}t$ and $x, z, t \in Y$. Because $|x| < |w|$, $|z| < |w|$ and $|t| < |w|$, the inductive hypothesis tells us that $x, z, t \in X$. By Part (1) of the definition on X, we have that $\% \in X$. Finally, since $\%, x, z, t \in X$, Part (3) of the definition of X tells us that $w = \mathtt{1}x\mathtt{0}z\mathtt{1}t = \%\mathtt{1}x\mathtt{0}z\mathtt{1}t \in X$.
  - Suppose $\mathbf{diff}(x) = 1$ and $\mathbf{diff}(y) = \mathbf{diff}(s) + 1$. Hence $\mathbf{diff}(y) = -1 + 1 = 0$, so that $y \in Y$. Since $\mathbf{diff}(x) \geq 1$, Lemma ES1.4.1 tells us that there are $z, t \in \{\mathtt{0},\mathtt{1}\}^*$ such that $x = z\mathtt{1}t$, $\mathbf{diff}(z) = 0$ and $\mathbf{diff}(t) = \mathbf{diff}(x) - 1$. Hence $z \in Y$ and $\mathbf{diff}(t) = 1 - 1 = 0$, so that $t \in Y$. Summarizing, we have that $w = \mathtt{1}s = \mathtt{1}x\mathtt{0}y = \mathtt{1}z\mathtt{1}t\mathtt{0}y$ and $z, t, y \in Y$. Because $|z| < |w|$, $|t| < |w|$ and $|y| < |w|$, the inductive hypothesis tells us that $z, t, y \in X$. By Part (1) of the definition on X, we have that $\% \in X$. Finally, since $\%, z, t, y \in X$, Part (2) of the definition of X tells us that $w = \mathtt{1}z\mathtt{1}t\mathtt{0}y = \%\mathtt{1}z\mathtt{1}t\mathtt{0}y \in X$.